An automaton describing left burnable configurations in the sandpile model on the ladder graph $\mathbf{Z} \times P_H$

Henri Derycke — Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France

Abstract

The sandpile model is well defined for any finite graph. Járai and Lyons approximated the behaviour on the bi-infinite ladder of height H in $\mathbb{Z}^2$ by a series of rectangles of height H and increasing width. This leads them in particular to the concept of left burnable configurations as a subset of recurrent configurations. We are interested in automata recognizing column by column these configurations. Járai and Lyons give an explicit automaton for the height $H = 2$ and have shown that there is an automaton for all H with rough bound $(2^H)^{2^H}$ over the number of states. Based on a bijection between the recurrent configurations and the spanning trees and on a combinatorial study, I propose a rough bound on the number of states in $\alpha^{H \log H}$. The algorithms linked to this bound make it possible to produce the automata for the height $H = 3$ with 13 states and $H = 4$ with 76 states. In this construction, the automaton contains at least β^H states because a subset of these states is in bijection with separable permutations (those avoiding 3142 and 2413). This work in progress is also linked and motivated by the analysis also in the course of simulations involving the sandpile model.

The Abelian sandpile model was introduced by physicists Bak, Tang and Wiesenfeld in [1] as a model of self-organized criticality. Given a simple, undirected graph $(V \cup \{s\}, E)$ where we distinguish s as the sink of the graph, we consider *configurations* in this model which are an assignments $\eta : V \mapsto \mathbf{N}$ of some grains of sand on each vertex. We say that η is stable at $x \in V$ if $\eta(x) < \deg(x)$, and η is stable if it is stable at all $x \in V$. If η is unstable at x, then x is allowed to topple which means that x sends one grain along each edge incident to it. Grains arriving at the sink are lost. Given a configuration η, we define a stabilization as a sequence of allowed toppling until a stable configuration is reached. The result of all stabilizations is unique due to commutation of toppling of unstable vertices and is noted $stab(\eta)$.

Let $P(\eta)$ be the result of a stabilization of $\eta + \mathbf{1}_{s\sim}$, that is η with an extra grain on each neighbour of s, which may be interpreted as a forced toppling of the sink. The set of recurrent configurations is a subset of the stable configurations characterized by Dhar [5] as the fixed points of P. In addition to that, each vertex topple exactly once in this process.

The notion of recurrence is related to a natural Markov chain in this model not discussed here [4], and it is well study for its connection with spanning trees [5], uniform spanning tree, the Tutte polynomial on the underlying graph [3, 9].

We are interested in the recurrent configuration on a family of ladder graph of height $H \geq 2$ defined as follow. Let $\Lambda_{W,H}$ be a finite graph derived from the ladder graph of $P_W \times P_H$, rooted on the left side to another vertex and with extra vertices on each line on the right side (Figure 1). A stable configuration on $\Lambda_{W,H}$ is a word on the alphabet $\{0, 1, 2\} \times \{0, 1, 2, 3\}^{H-2} \times \{0, 1, 2\}$ counting the number of grains on a column's vertices.

In this paper, we present the construction of an automaton that accepts all the recurrent configurations of the graphs $\Lambda_{\bullet,H}$. The automaton will respect the same properties than the description given by Járai and Lyons in [7] while improving the rough bound on states from $(2^H)^{2^H}$ to $2^H H!$. Gamlin suggest the existence of an automaton with at most α^H states [6, Appendix] as for spanning trees. This approach is related to the dimer model (see [8]). However, the bijections with spanning trees are non local, then we may use the same transfer matrix as trees with an

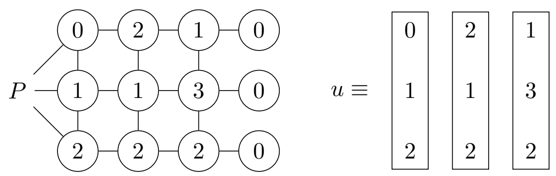

Figure 1: A recurrent configuration on $\Lambda_{3,3}$

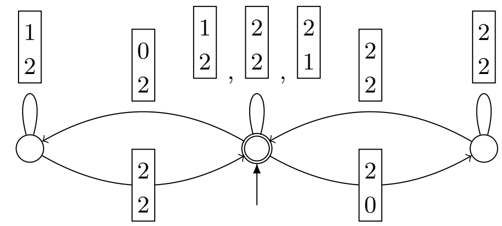

Figure 2: Automaton for $H = 2$

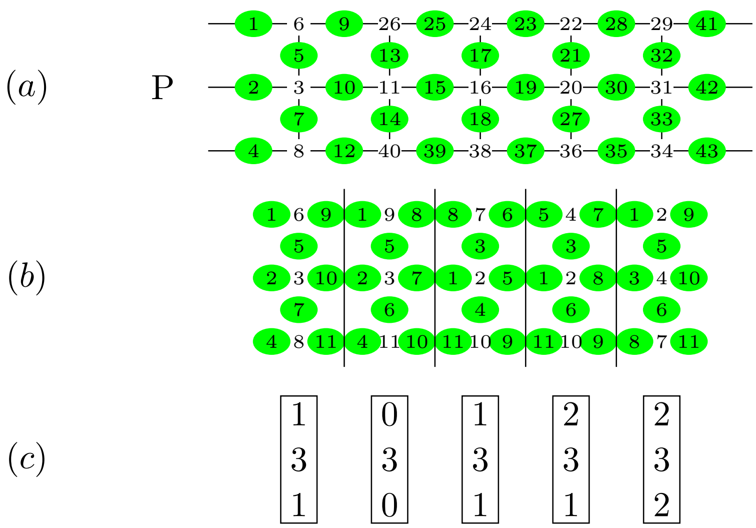

Figure 3: Column decreasing decomposition and configuration

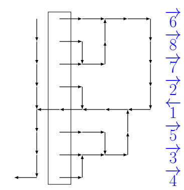

Figure 4: Tree and separable permutation

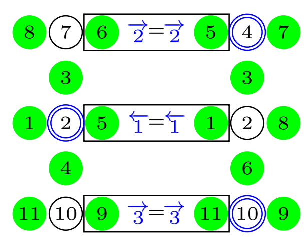

Figure 5: Compatibility rules

additional information that seems to force the $H!$. This construction is based on *decreasing edge-vertex traversal* introduced by Cori and Le Borgne [3] in a bijection between spanning trees and recurrent configurations (turning Tutte externally active edges into excess of grains).

Let $<_e$ be an order on E. An *edge-vertex traversal* $\sigma = (\sigma_i)_{1 \leq i \leq |V|+1+|E|}$ of $G = (V \cup \{s\}, E)$ is a permutation over the $V \cup \{s\} \cup E$. We note $\sigma^{<i} = \{\sigma_j \mid j < i\}$. A *decreasing edge-vertex traversal* is an edge-vertex traversals such that (i) if σ_i is a vertex of V, then σ_{i-1} is an edge incident to σ_i, (ii) if σ_i is an edge, it is minimal to $<_e$ among all the edges not in $\sigma^{<i}$ and incident to a vertex of $\sigma^{<i}$.

Lemma 1 (Cori,Le Borgne[3])**.** *The recurrent configurations and the decreasing edge-vertex traversals are in bijection.*

The bijection can be seen as the process of the fixed point characterization by Dhar, treating each grain one after the other following the order over the edges. Corollary, the number of grains of a vertex x is given by the number of edges incident to x that appear after x in σ. And, the spanning tree is deduced from the set of edges that precede a vertex in the traversal. Figure 3 (a) illustrates an example of decreasing traversal. Each vertex and each edge (in green) is labelled by its position in σ.

Let $<_e$ be the order on the edges of $\Lambda_{\bullet,H}$ from left to right and top to bottom. We note D_H the set of decreasing traversals on the $\Lambda_{\bullet,H}$. Let $\sigma \in D_H$ a decreasing traversal on $\Lambda_{W,H}$, we decompose σ in a sequence $(c_i)_{1 \leq i \leq W}$ of *column decreasing traversals* where c_i is the subsequence of σ on the vertices of the i-th column of $\Lambda_{W,H}$ and the edges incident to them (Figure 3 (b)).

Property 1. *The set of decreasing traversals and their decomposition in column decreasing traversals are in bijection.*

Proof. Let $(c_i)_{1 \leq i \leq W}$ be a decomposition. Each c_i contains the order of the vertices of the $i - th$ column and its incident edges. Thus from the corollary, we have the number of grains in each vertices. The bijection of Cori-Le Borgne provides the original decreasing traversal. □

Let C_H be the set of the column decreasing traversals that appears in the decomposition of some $\sigma \in D_H$.

Theorem 1. *A* $(c_i)_{1 \leq i \leq W} \in (C_H)^W$ *is the decomposition of decreasing traversal if and only if:*

1. *the first edges in* c_1 *are the* H *left horizontal edges from top to bottom.*
2. *the last elements of* c_1 *are the* H *right horizontal edges from top to bottom.*
3. *for all* $1 \leq i < W$*, the order in* c_i *et* c_{i+1} *of the* H *horizontal edges incident both to columns* i *and* $i + 1$ *respect:*
   - *(a) the orders in* c_i *and* c_{i+1} *are the same*
   - *(b) for each edge, if it appears before its end in* c_i*, it appears after its end in* c_{i+1} *and vice versa.*

The item 3. of theorem 1 is compatibility rule illustrated by Figure 5. Let $\sigma \in D_H$. For each horizontal edge $e = x \sim y$, either $\sigma = \ldots x \ldots e \ldots y \ldots$ or $\sigma = \ldots y \ldots e \ldots x \ldots$. The first endpoint is marked with a double circle. The blue numbers with arrows hat are the information that describes compatibility. Given $c \in C_H$, we can derive two vectors $left(c)$ and $right(c)$ describing for each side the order of its horizontal edges and there orientation toward their first endpoint.

Sketch of the proof. First, we show that these are necessary conditions. The condition 1 and 2 are direct consequences of the construction of [3]. The edges incident to the sink are the first in a decreasing traversal and the last edges are the edges incident to the extra vertices on the right. Condition $3a$ is straightforward. From [3] we have that an edge appears after its endpoints if and only if it is externally active in the tree, that is, it is maximal according to $<_e$ in its induced cycle. One can show that the order $<_e$ guarantees that the maximal edge of any cycle is vertical. Then, an horizontal edge is not externally active so, it is not after its endpoints.

In order to show that the conditions are sufficient is based on the completion of an linear extension step by step looking between two consecutive columns. The horizontal edges split the graph in two disjoint part. Then the order on the left is not relevant for the order on the right as long as the order of the shared edges is the same. □

The theorem 1 provides the description of an automaton with alphabet C_H. For each $c \in C_H$, we have exactly one transition from $right(c)$ to $left(c)$. Then the states of the automaton are the set of $left(c)$ and $right(c)$ for $c \in C_H$.

The number of states is bound by $2^H H!$ and the number of transitions is $|C_H|$. The automaton is derived from C_H. However we don't have a combinatorial description of D_H yet. In order to

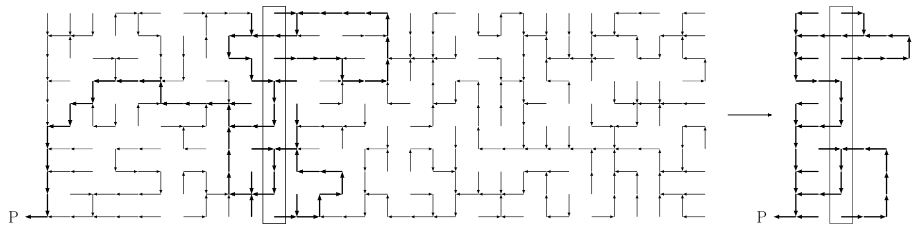

Figure 6: Simplification from a generic tree

construct the automaton, we enumerate D_H from the spanning trees of a graph $\Lambda_{W,H}$ for $W \leq 2H+3$

Property 2. *The set C_H is also the set of the column decreasing traversal that appears in the decomposition of the $\sigma \in D_H$ of $\Lambda_{2H+3,H}$.*

Sketch of the proof. Given a $c \in C_H$ and a tree T that produces c, it's possible to apply transformations on T that preserve c (Figure 6). The tree T is pruned from the branches that don't lead to a vertex of c or a adjacent vertex of c. Then the branch on the left can be reduced to an imbrication of combs. The transformation on the right has to preserve the order of the branches according to their maximal edges. The resulting tree needs at most H columns on the left and H columns on the right. □

As a by product, this construction gives an injection from separable permutations [2] to a subset of states ans a exponential lower bound on the states. A separable permutation can be defined by a binary tree that can be embedded in $\Lambda_{\bullet,H}$ as in Figure 4.

From this automaton, we have an automaton on the configurations by projection. However, this projection loses the lower and upper bound.[1]

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[1] see `https://www.labri.fr/perso/hderycke/ladder_automata.html` for the automata for height 3 and 4