An automaton describing left burnable configurations in the sandpile model on the ladder graph

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LaBRI

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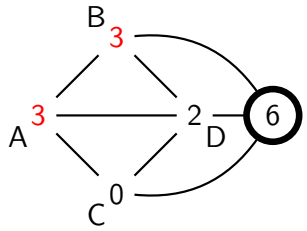
université de BORDEAUX

The sandpile model



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$G = (V, E)$ simple connected graph with a special vertex $S \in V$ called the *sink*

- Configuration : assignment $u : V \to \mathbf{N}$ of grains on the vertices
- A vertex distinct from the sink is *unstable* if $u(x) \geq \deg(x)$
- Toppling : a vertex sends one grain to each of its neighbors
- Stabilization : While there is an unstable vertex, we topple it. Given u, we note $\mathrm{stab}(u)$ the result.

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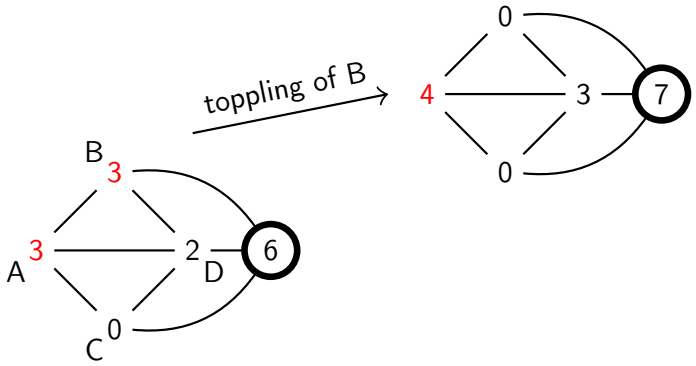

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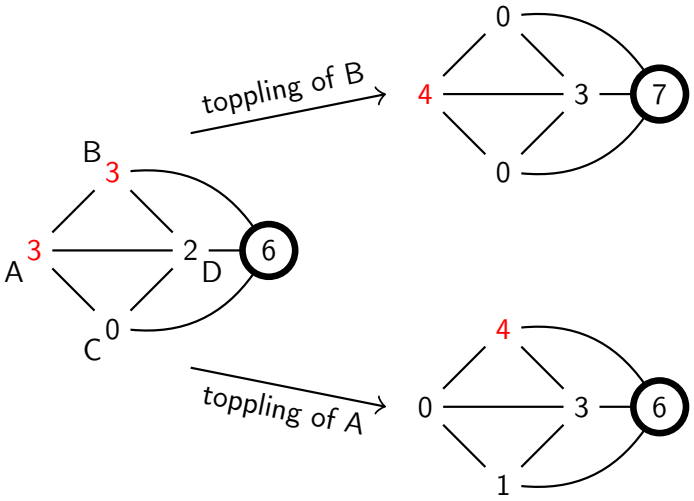

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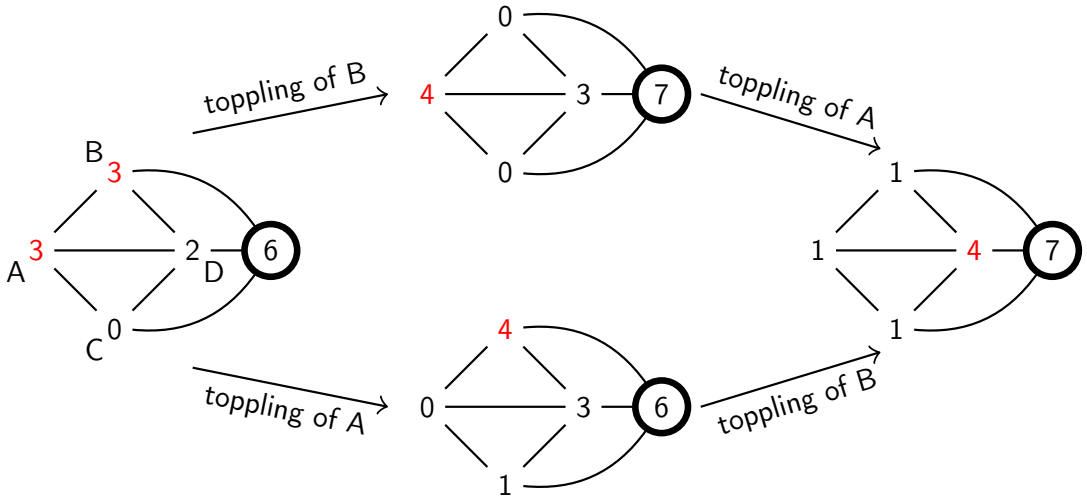

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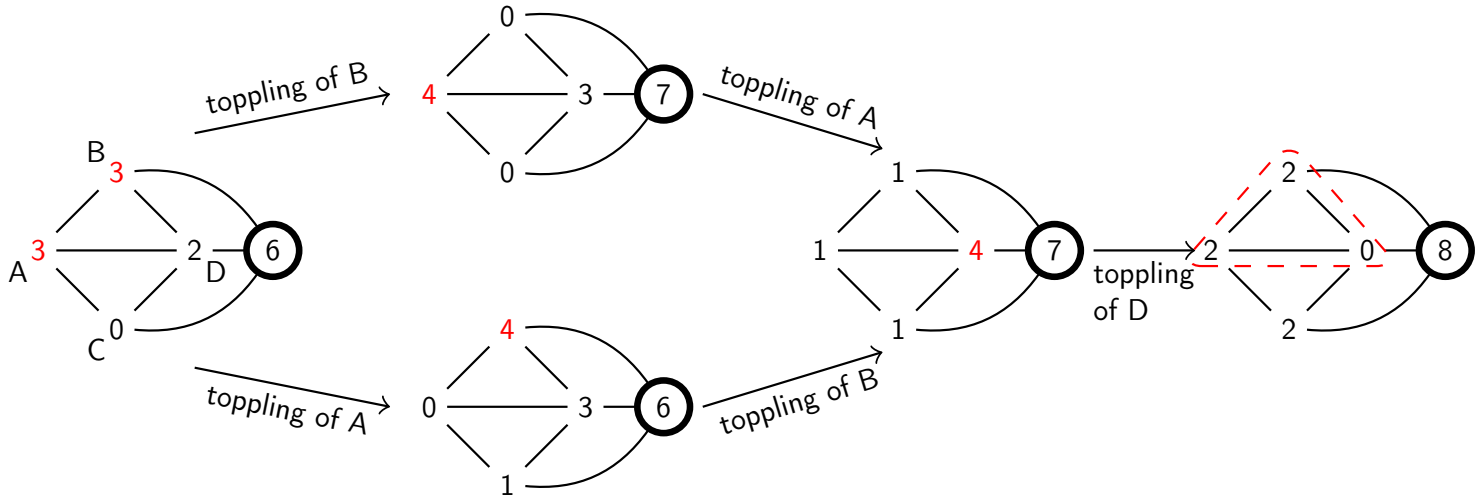

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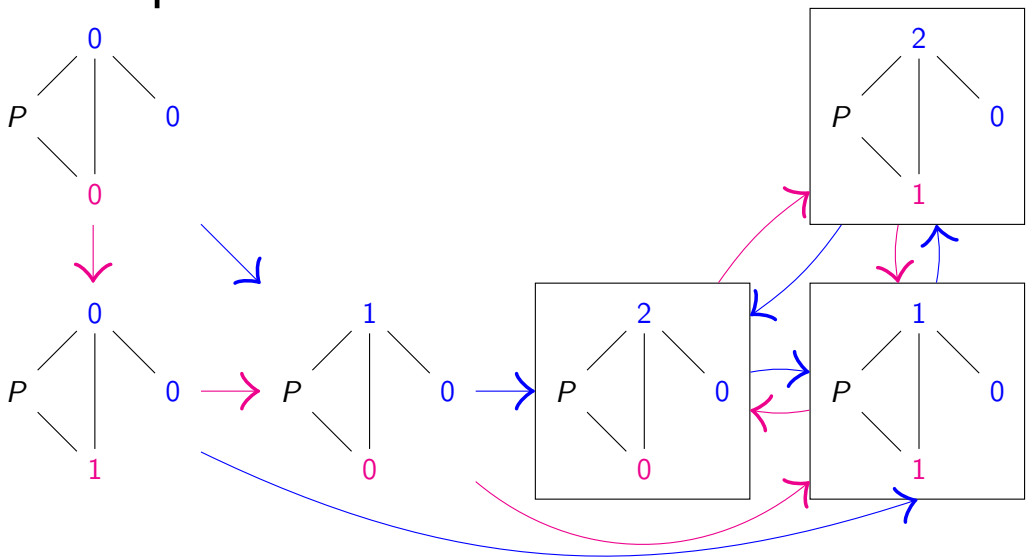
Markov Chain for $G = (V \cup \{S\}, E)$

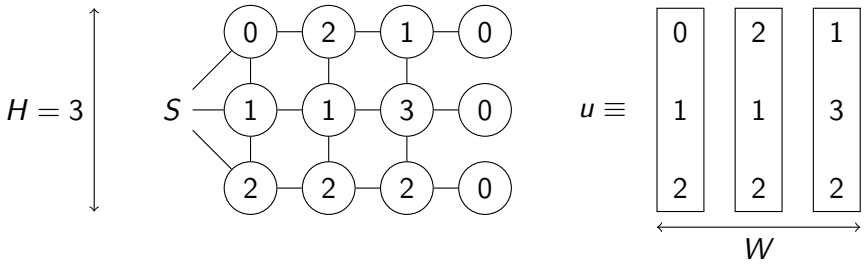
- States : stable configurations on G
- Transition : Add a particle to a vertex chosen uniformly and stabilize



- The recurrent states are called recurrent configurations.
- The stationary distribution is uniform on the recurrent configurations.

Dhar Criterion A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)

Configurations on left rooted square grid graphs

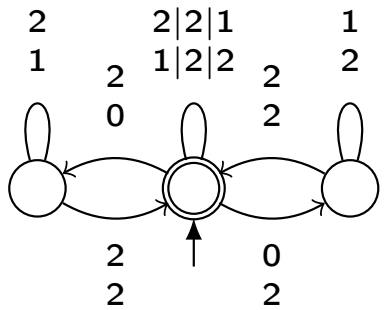


A configuration u may be read as a word of size W on the alphabet of columns.

Járai, Lyons 07

The recurrent configurations of the left-rooted square grid graphs are recognized by a finite automaton on the columns.

For height 2, they found the following automaton.



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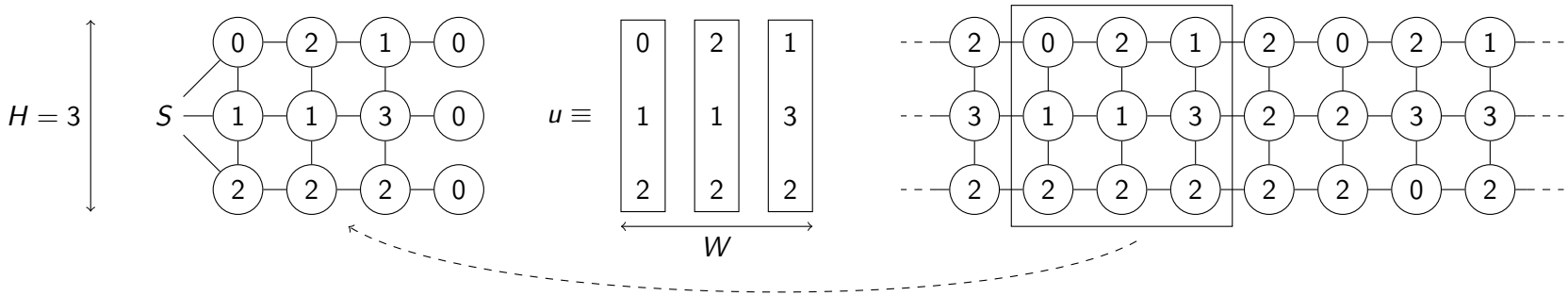

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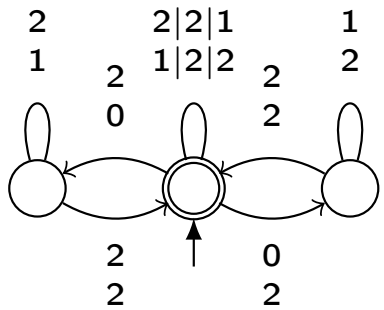


Some results on these automata of height $H \geq 2$

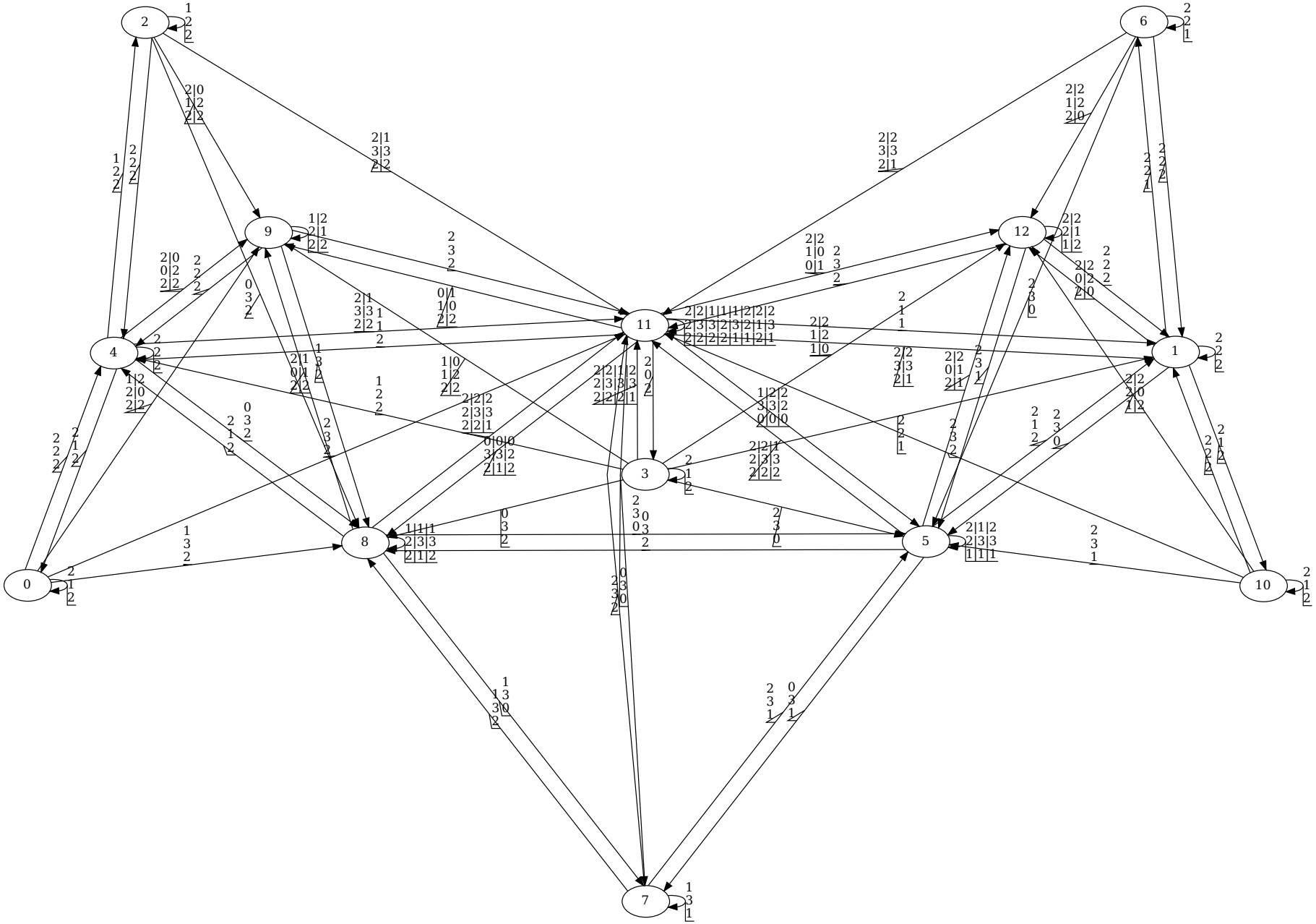
From Járai and Lyons, the number of states is roughly bounded by $(2^H)^{2^H}$.

Conjecture from Gamlin (Thesis 2015) $\exp cH$ states may be enough.

Similar to an automaton for spanning trees (eg. dominoes tiling) but optimistic on the non local transformation to recurrent configurations.

My results

- Efficient construction of an automaton.
- From α^H to $\beta^{H \log H}$ states.
- The lower bound is from a bijection between a subset of the states and separable permutations (avoiding patterns 2413 and 3142).
- Explicit automata for heights 3 and 4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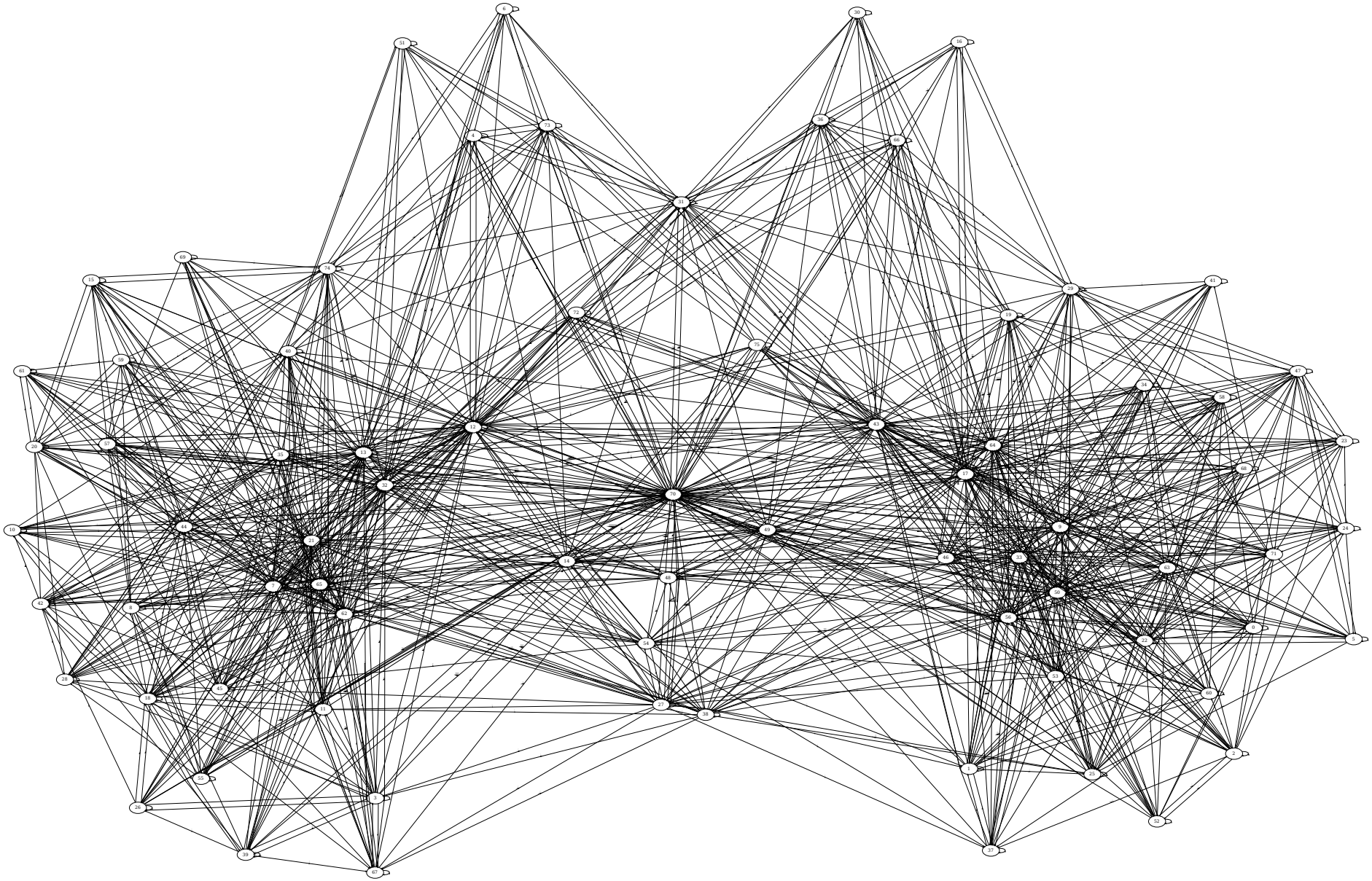

Bijections between recurrent configurations and spanning trees

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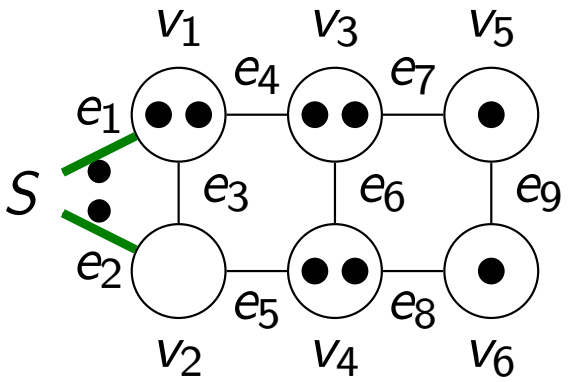
Bijection CLB : (from a recurrent configuration)

Decreasing traversals : $e_1, v_1, e_2, e_3, e_4, v_3, e_6, v_4, e_5, v_2, e_7, v_5, e_8, v_6, e_9$

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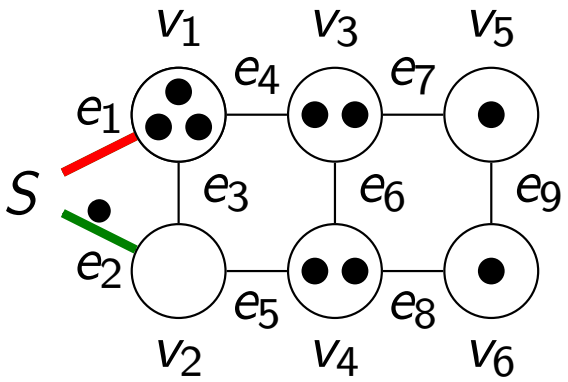


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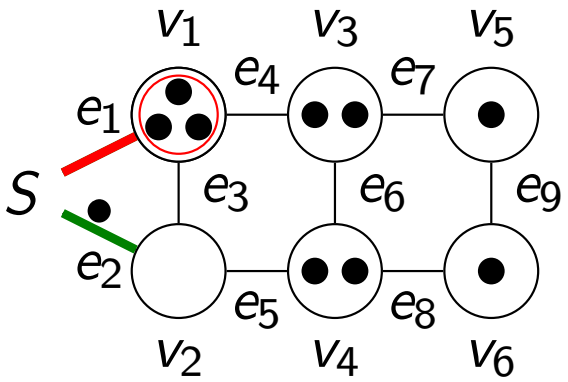


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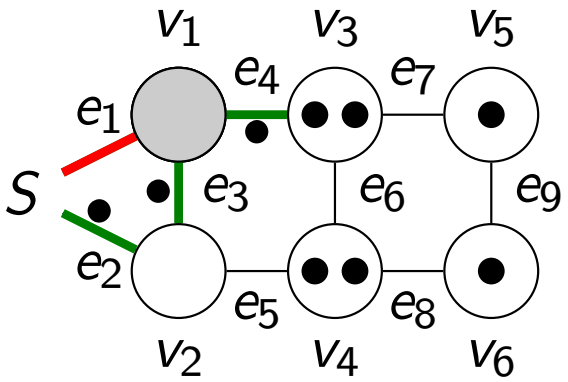


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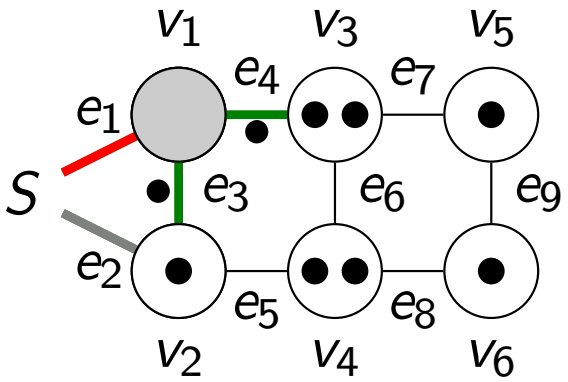


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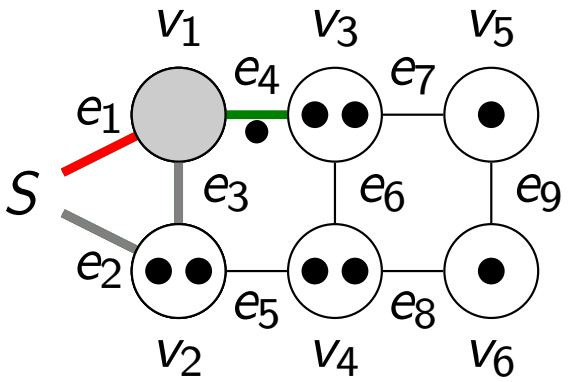


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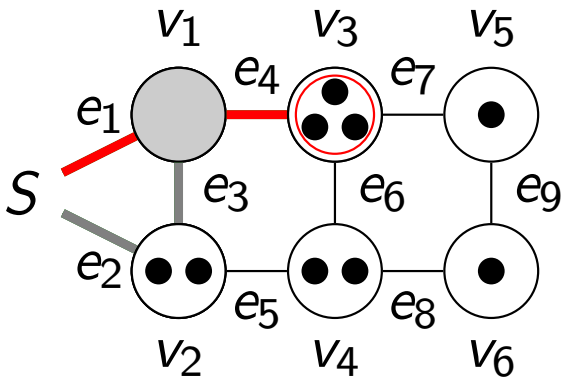


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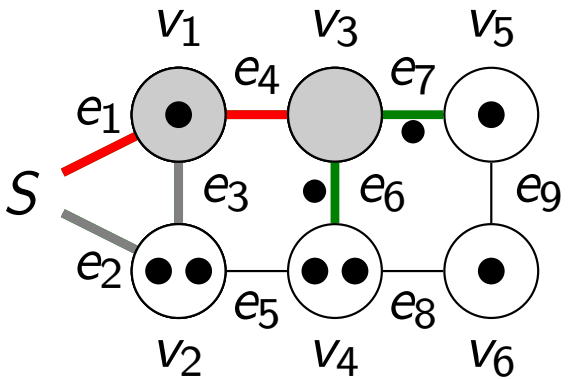


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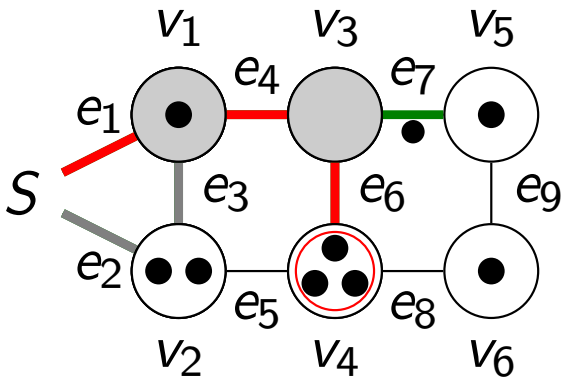


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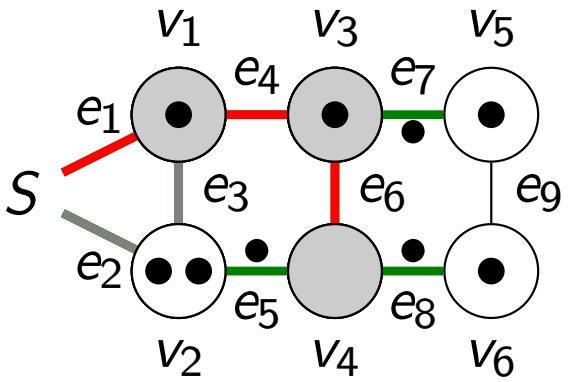


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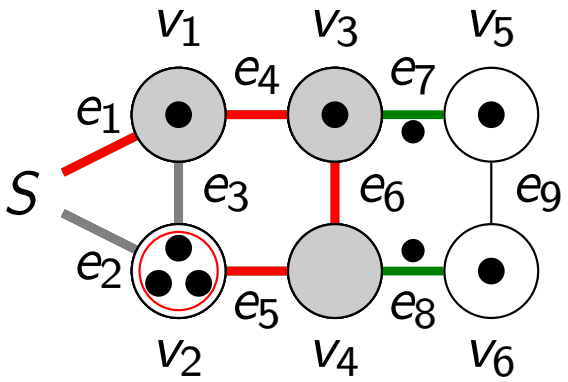


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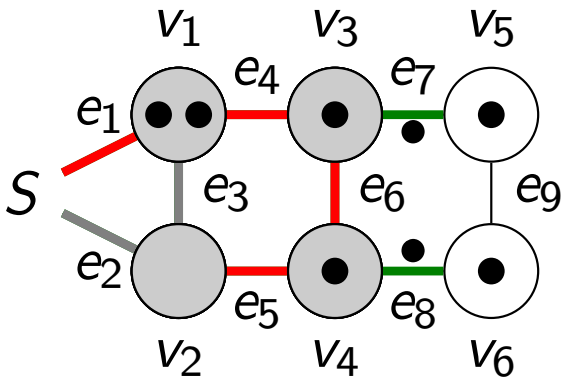


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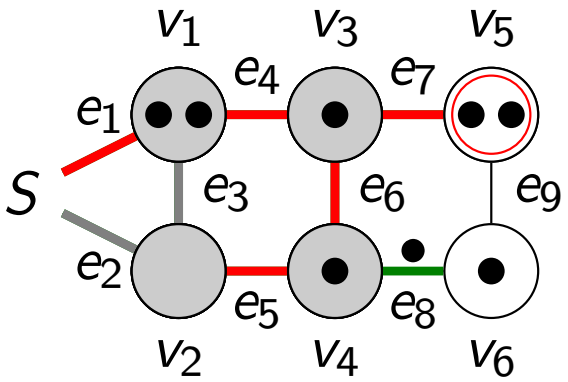


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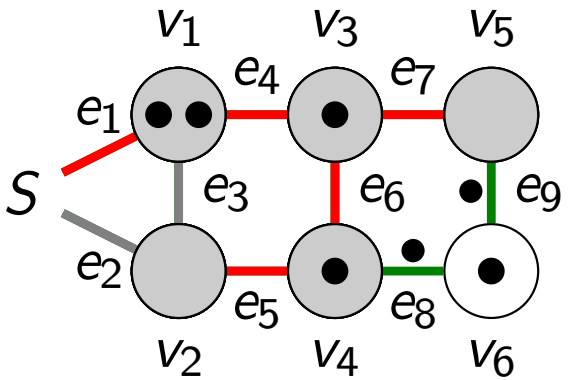


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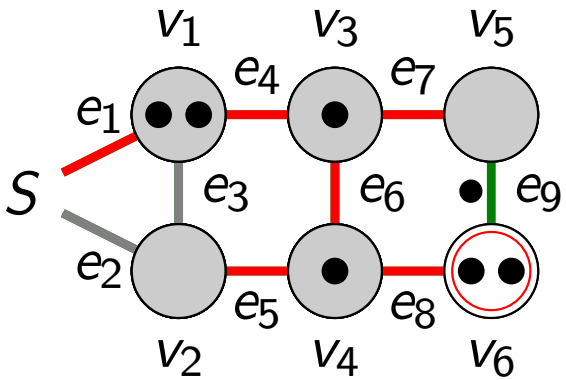


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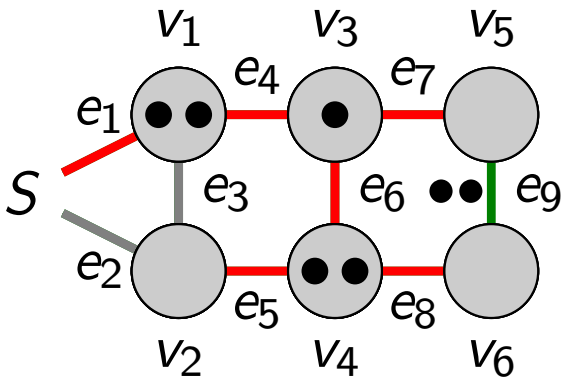


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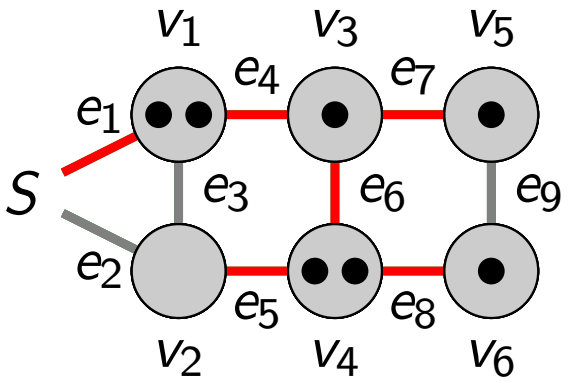


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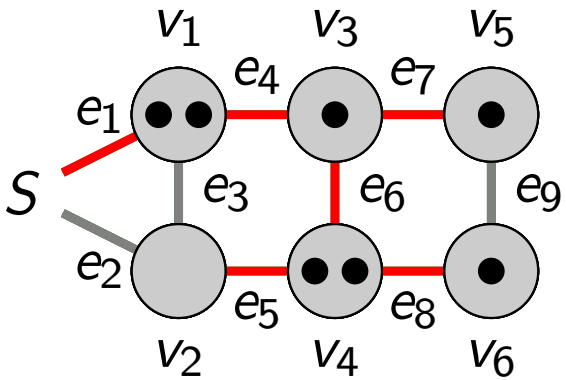
We choose an order on the edges : from left to right and from top to bottom.

Bijection CLB : (from a recurrent configuration)

Decreasing traversals : $e_1, v_1, e_2, e_3, e_4, v_3, e_6, v_4, e_5, v_2, e_7, v_5, e_8, v_6, e_9$

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Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the smallest pending edge
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
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- The spanning tree is encoded by the red edges that precede vertices in the decreasing edge-vertex traversal.
- For any vertex, the number of incident edges that are treated after the vertex is equal to the number of grains in the recurrent configuration.

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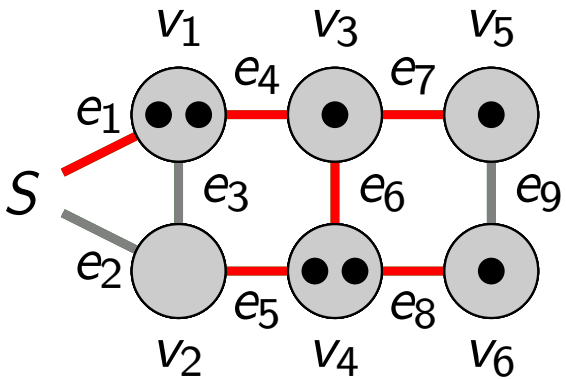


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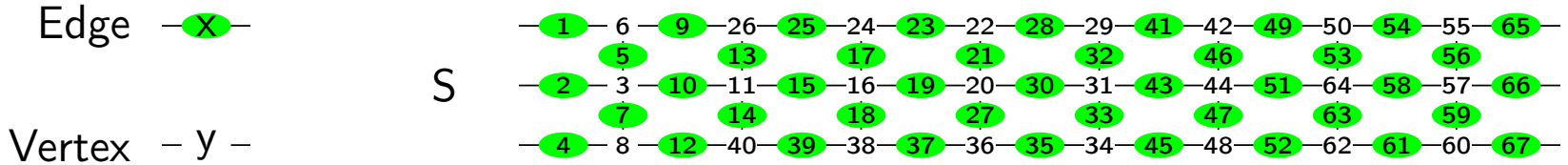
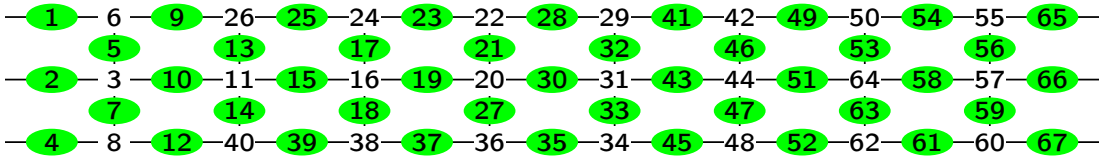

Edge
X
Vertex
y
S
1 6 9 26 25 24 23 22 28 29 41 42 49 50 54 55 65
5 13 17 21 32 46 53 56
2 3 10 11 15 16 19 20 30 31 43 44 51 64 58 57 66
7 14 18 27 33 47 63 59
4 8 12 40 39 38 37 36 35 34 45 48 52 62 61 60 67

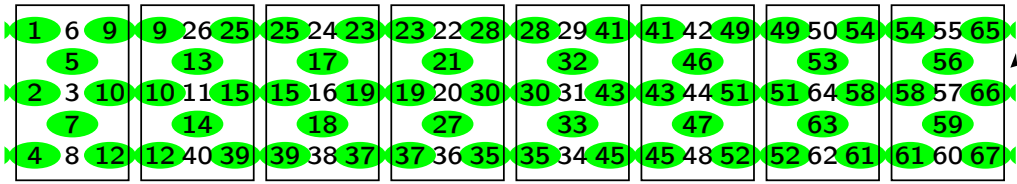
Edge – x –

Vertex – y –

S



- Decomposition



Edge – x –

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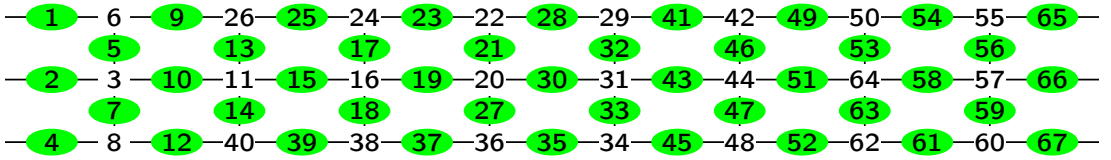

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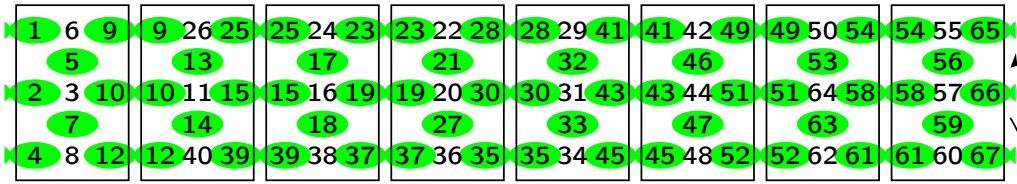
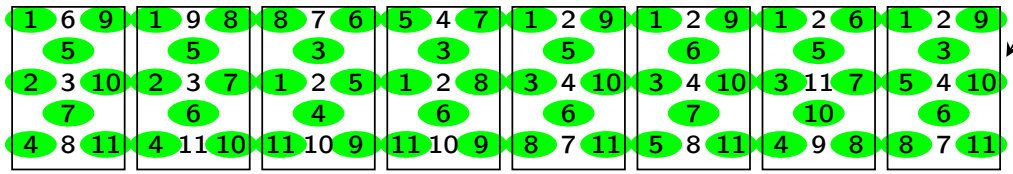
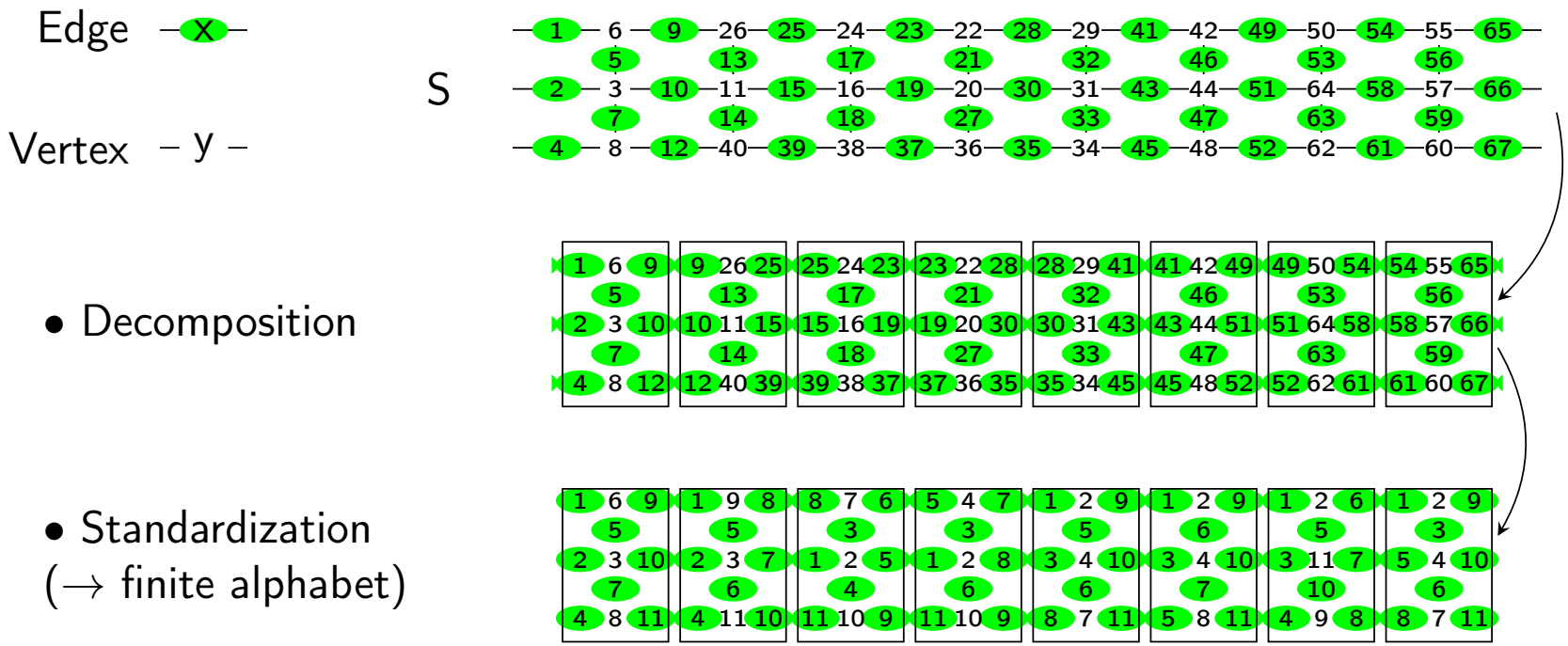


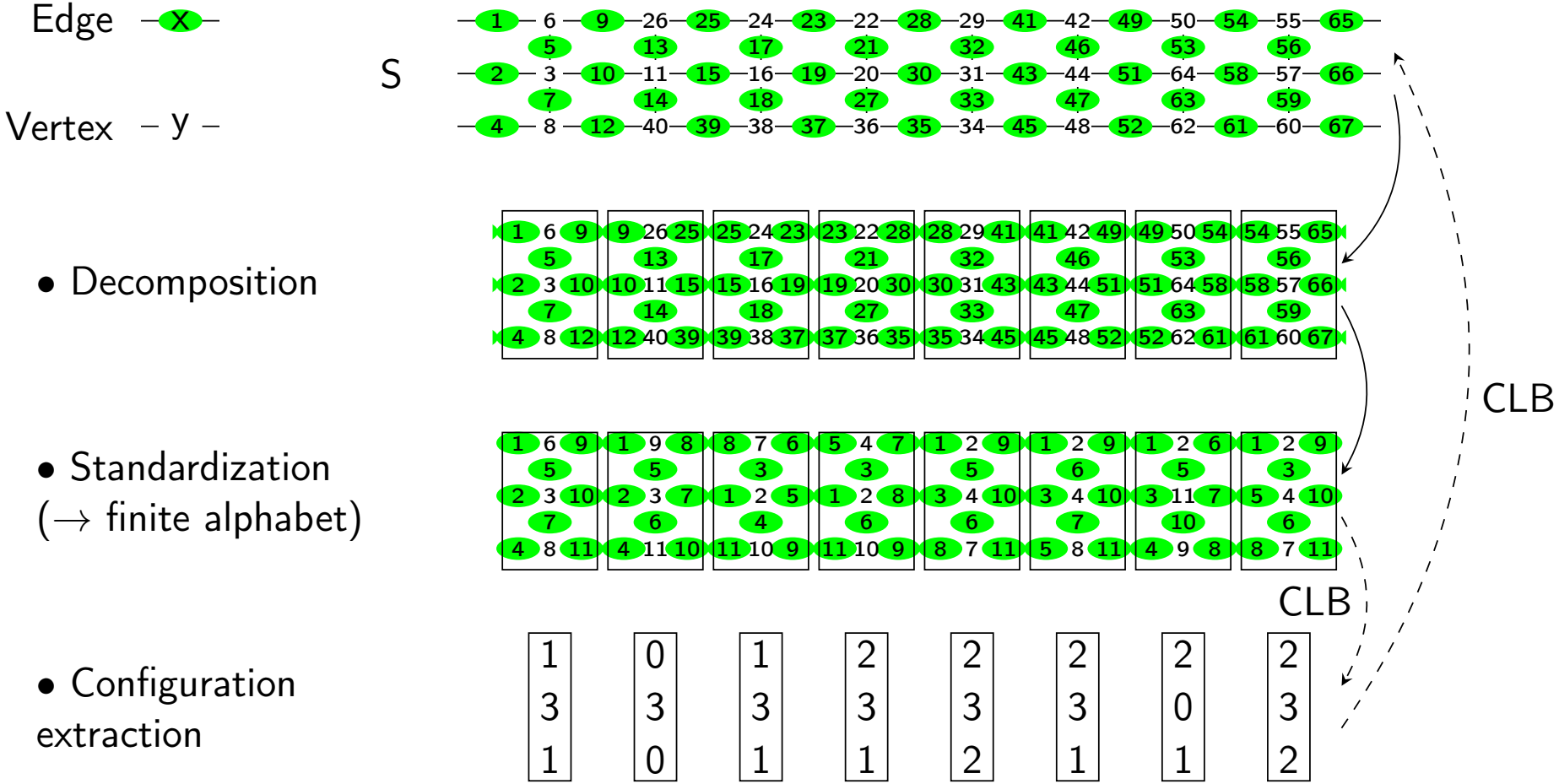
- Standardization
($\rightarrow$ finite alphabet)





Proposition

The edge-vertex decreasing traversals are in bijection with their standardized decompositions (in induced orders).



- Decomposition
- Standardization ($\rightarrow$ finite alphabet)
- Configuration extraction

Proposition

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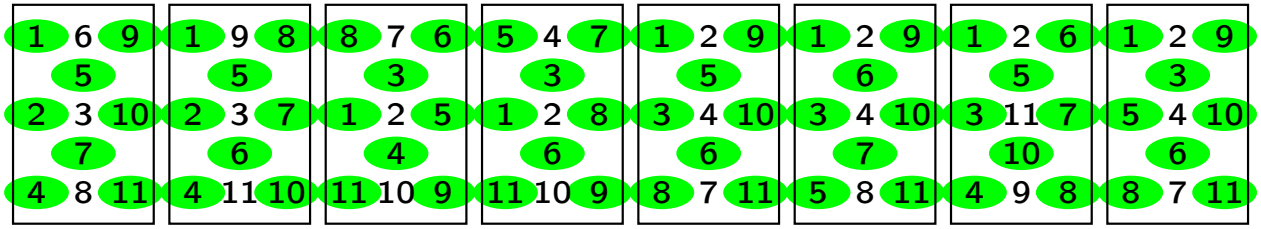
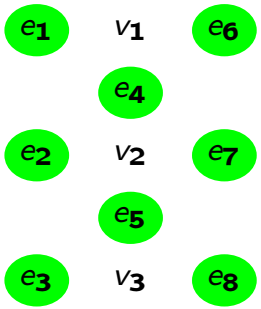


A decomposition is a word on the alphabet of induced orders.



- Initial induced order : $e_1(v_1?)e_2(v_2?)e_3(v_3?)\dots$
- Sequence of consecutive pairwise compatible induced orders : (o, o') verify $T(o, o')$
- Final induced order : $\dots e_6 e_7 e_8$

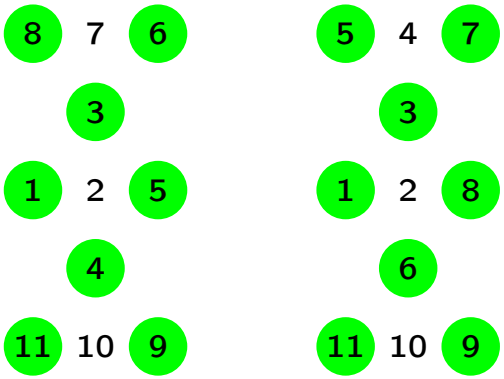
Automaton on induced orders

- States : induced orders
- Alphabet : induced orders
- Transition : $o \xrightarrow{o'} o'$ iff o' is right-compatible to o

Prop $\sharp stats \leq (4H - 1)!$

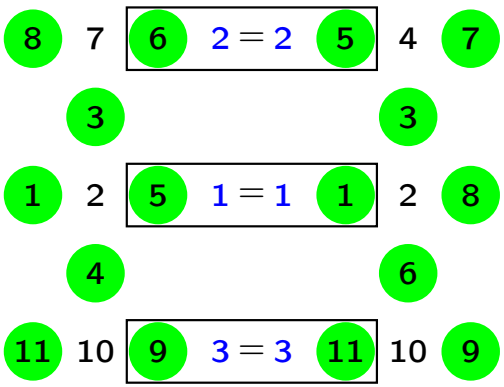
We have an automaton recognizing recurrent configurations

Right compatibility relation



Necessary conditions

Right compatibility relation



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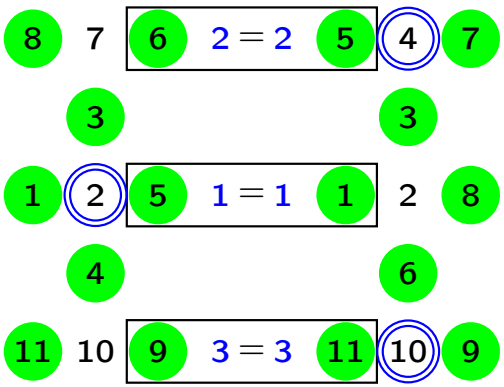

Necessary conditions

- the orders on common edges are the same
- horizontal edges are after at least one of their endpoints

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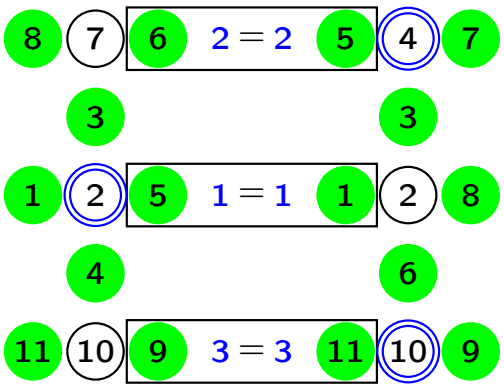

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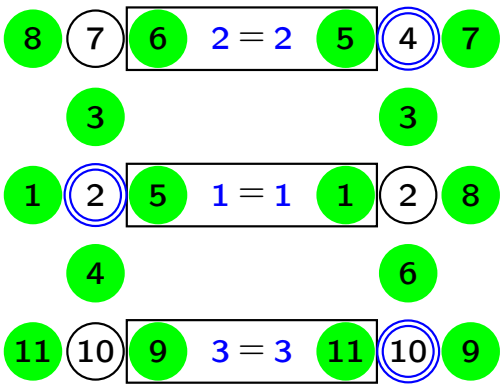

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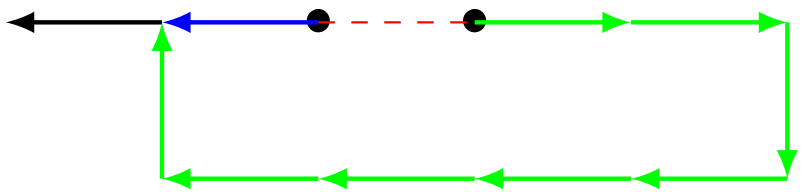

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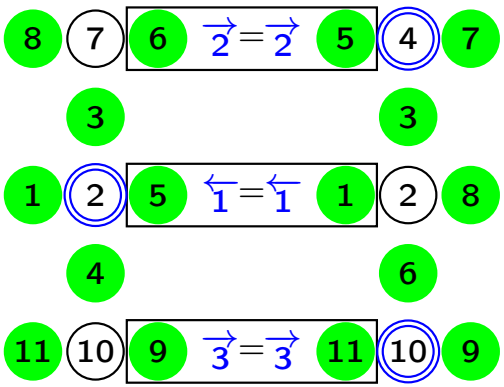

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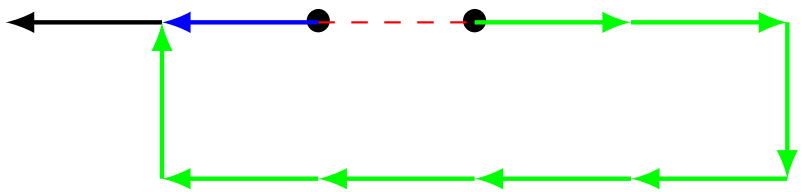

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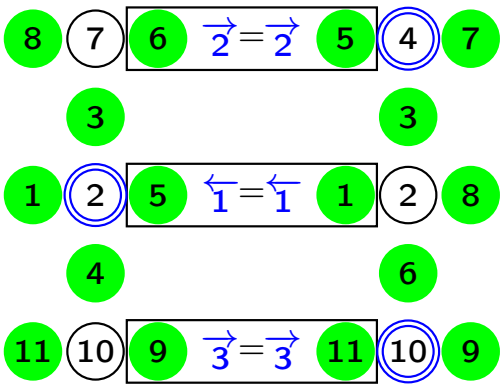

Necessary and sufficient conditions

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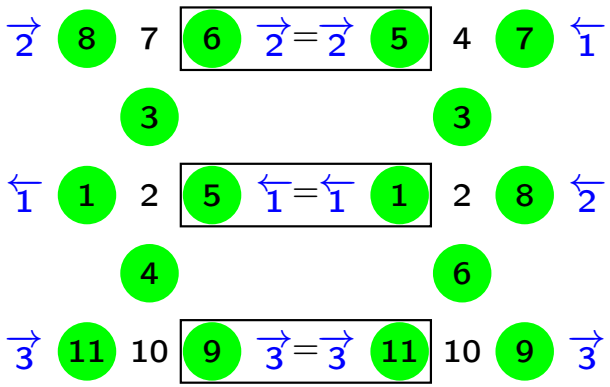

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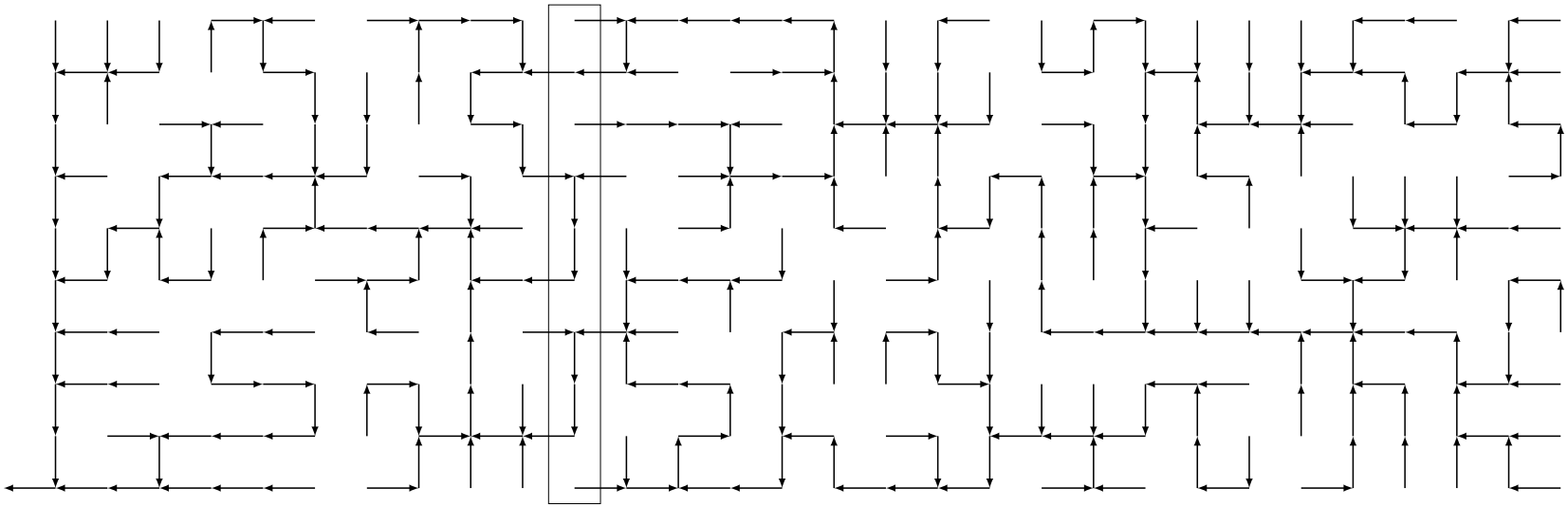
Reduction on the number of states : permutations and orientations on the right horizontal edges

- $\sharp states \leq 2^H \cdot H! (\leq (4H-1)!)$
- the transitions and alphabet are still the induced orders

Proposition

Any induced order appears as a letter on some recurrent configuration's standardized decomposition on $[1, W] \times [1, H]$ for $W = 2H + 1$.



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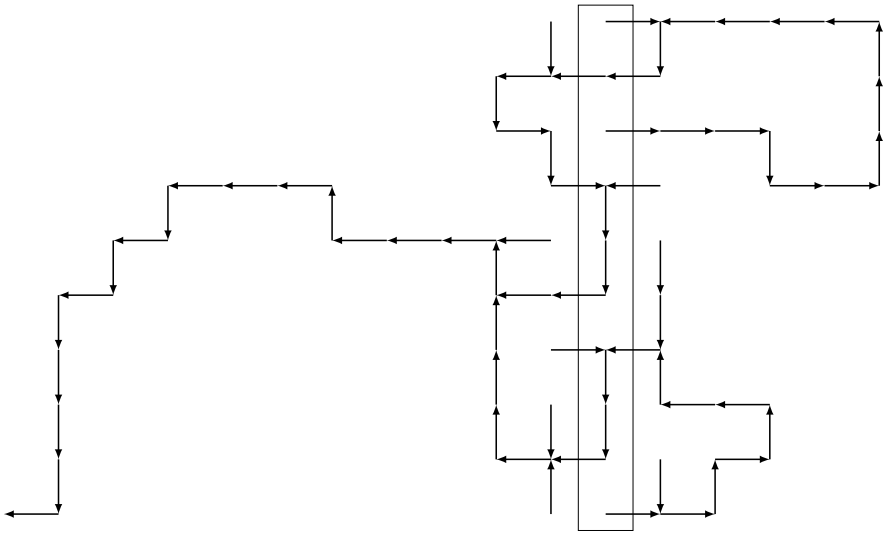

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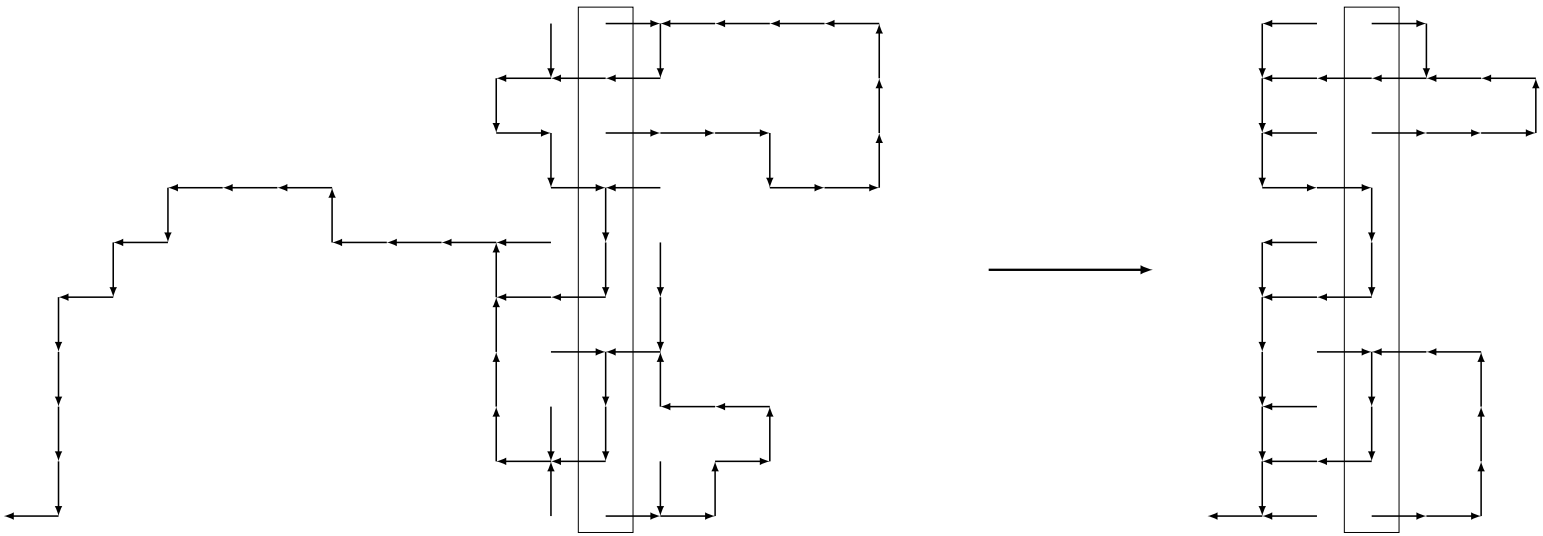

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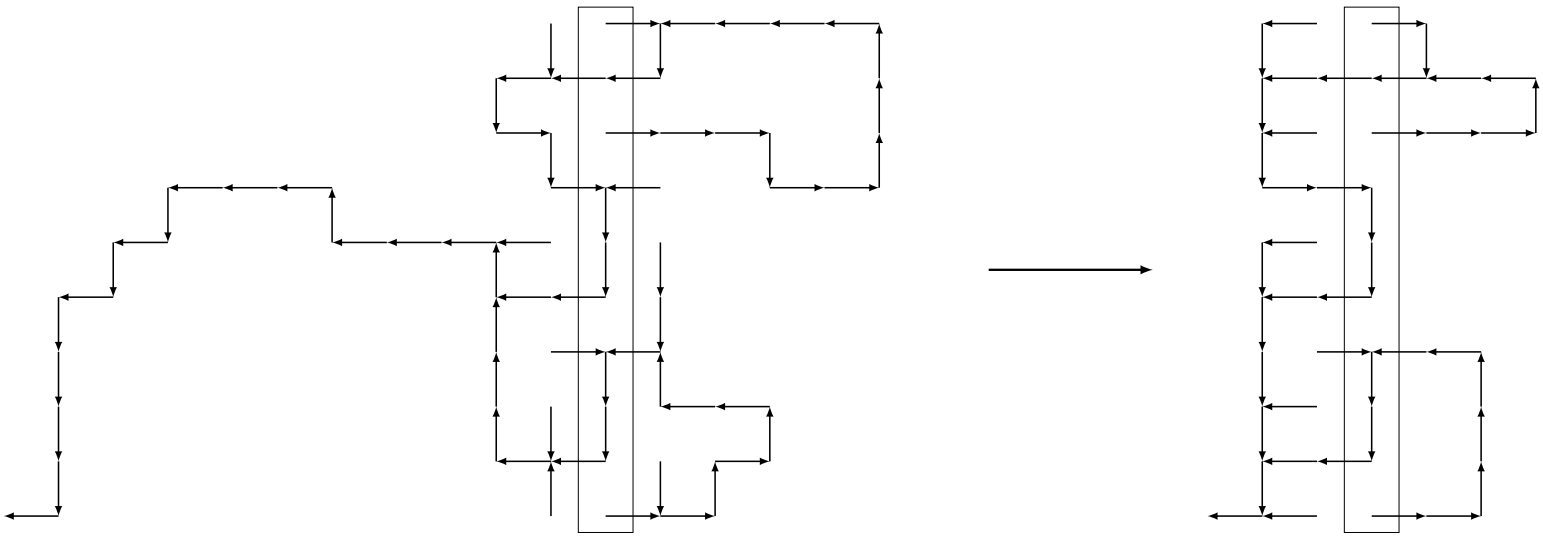
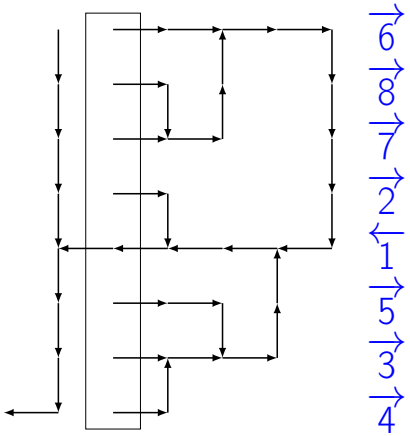
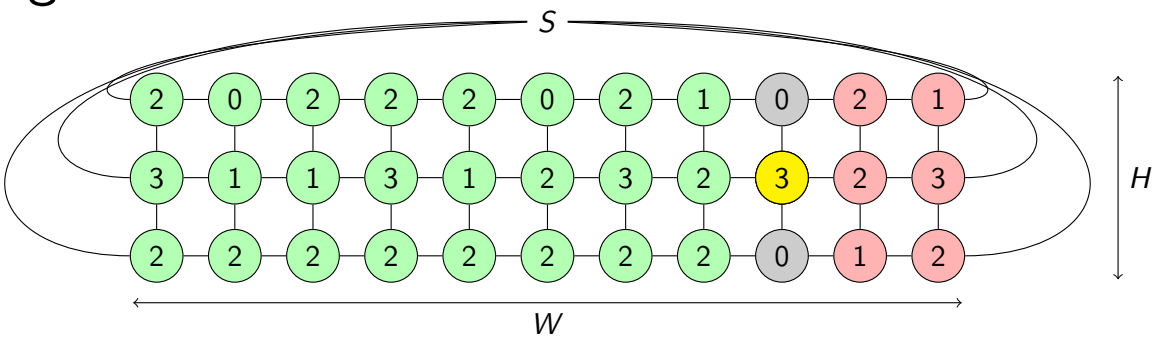




- Order over the right horizontal edges : $5, 4, 7, 8, 6, 1, 3, 2$
- Binary trees $\Rightarrow$ Separable permutations (avoiding pattern 2413 and 3142)
- Exponential lower bound on the number of states

Towards the infinite ladder (Járai, Lyons 07)

Approximation of $\mathbf{Z} \times [1, H]$ by square grid graphs rooted to the sink on the left and right sides.



- vertices that topples from the left (green and yellow)
- vertices that topples from the right (red and yellow)
- other vertices (gray)

A *left burnable configuration* is a green/yellow configuration.
Decomposition in left-burnable blocks..

Automaton on a super set of column alphabet $\Rightarrow$ Parry measure $\Rightarrow$ Uniform distribution on left-burnable configurations of $\mathbf{Z} \times [1, H]$

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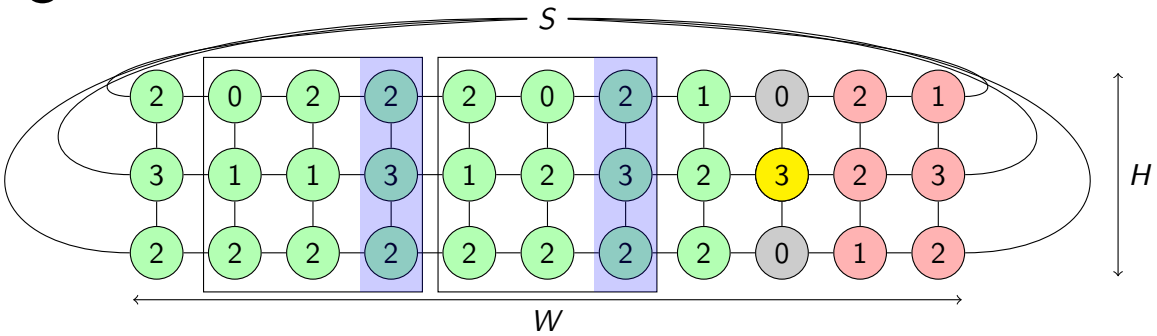

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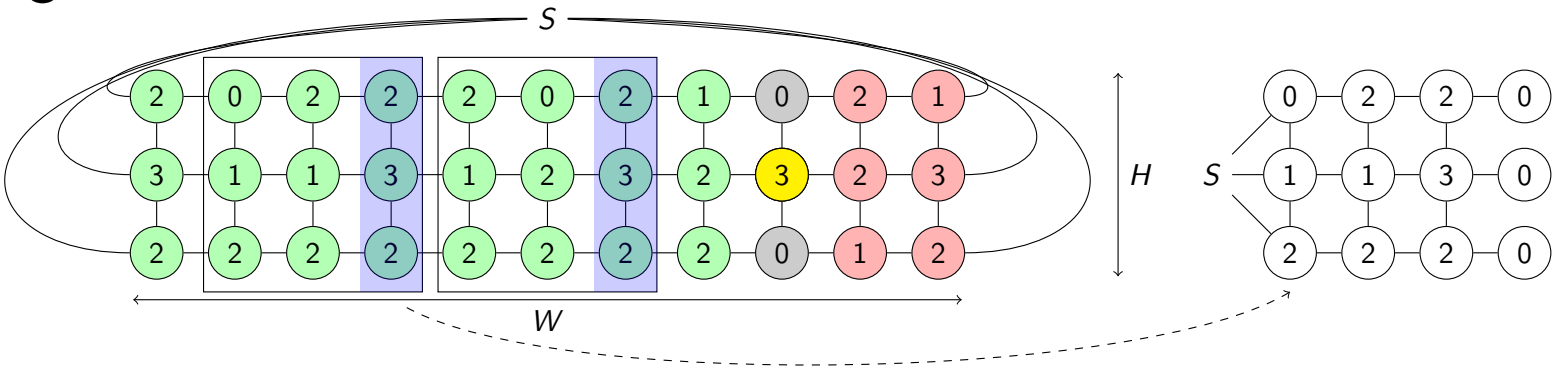

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