Une notion de récurrence dans le modèle du tas de sable sur le réseau carré

Henri Derycke  
joint work with Yvan Le Borgne

LaBRI

JCB 2019, Février 11-13, Bordeaux
Configuration: $\eta : V \mapsto \mathbb{N}$

$v \in V$ is unstable for $\eta$ if $\eta(v) \geq \deg(v)$, it is stable otherwise.
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$\nu \in V$ is unstable for $\eta$ if $\eta(\nu) \geq \operatorname{deg}(\nu)$, it is stable otherwise.

Toppling $u : \eta \mapsto \eta + \Delta^{(u)}$ If $u$ is unstable, the toppling is legal. It is forced otherwise.
Sandpile Model [Bak, Tang, Wiesenfeld 87]

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The order of toppling does not change the result: $\eta \rightarrow \eta + \sum_{v \in V} a_v \Delta^{(v)}$. 
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Stabilisation: while a vertex is unstable, topple it.
The sink

How to stabilize (even with a large number of grains)?

\[
\begin{array}{cc}
2 & 2 \\
3 & 3 \\
2 & 2 \\
\end{array}
\]
The sink

How to stabilize (even with a large number of grains)?

We distinguish a vertex as the sink that won’t topple.

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\begin{array}{cc}
2 & 2 \\
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The sink guarantees that the stabilisation of any configuration $\eta$ terminates and we note the result $\mathsf{stab}(\eta)$.
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```
<table>
<thead>
<tr>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
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</tr>
</tbody>
</table>
```

Stabilisation

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
</tbody>
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Markov Chain

- States: stable configurations on $G$
- Transition: Add a particle from the sink to a vertex chosen uniformly and stabilize

Recurrent states are in the same connected component.
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**Markov Chain**

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Test of recurrence

Dhar operator

Topple the sink (forced), then stabilize: \( \text{dhar}(\eta) := \text{stab}(\eta + \Delta^{(s)}) \)
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\]

\[\begin{array}{cc}
0 & 1 \\
2 & 2 \\
s & 1 \\
\end{array}\quad \rightarrow \quad \begin{array}{cc}
0 & 1 \\
3 & 2 \\
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\end{array}\]
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\[
\begin{array}{cc}
0 & 1 \\
2 & 2 \\
\text{s} & 1 \\
\end{array}
\quad \rightarrow 
\begin{array}{cc}
0 & 1 \\
\text{3} & 2 \\
\text{s} & 2 \\
\end{array}
\quad \rightarrow 
\begin{array}{cc}
1 & 1 \\
0 & 4 \\
\text{s} & 0 \\
\end{array}
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A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.
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2 & 2 & 3 \\
s & 1 & s \\
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0 & 1 & 0 \\
3 & 2 & 4 \\
s & 2 & s \\
\end{array} \quad \rightarrow \quad \begin{array}{c|c|c}
1 & 1 & 0 \\
0 & 4 & 1 \\
s & 0 & s \\
\end{array} \quad \rightarrow \quad \begin{array}{c|c|c}
1 & 2 & 1 \\
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s & 1 & s \\
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\hline
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\begin{array}{cc}
0 & 1 \\
2 & 2 \\
s & 1 \\
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3 & 2 \\
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Bijections with spanning trees

**Theorem (Dhar, Majumdar 92)**

*The recurrent configurations for a finite graph G and its spanning trees are in bijection.*

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce’s path for sorted recurrernts on $K_n$)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

**Cori/Le Borgne 03 (CLB)** (e.g. Dhar criterion more uniform in space)

Mark edges incident to the sink as pending edges.

While there is a pending edge

- Get the closest pending edge to the sink
- Process the grain(s) on the edge
- If a vertex become unstable, topple it and mark its untreated incident edges as pending edges.

Edge-vertex traversal: $s$, 

$v_1 \quad e_1 \quad v_2 \quad e_2 \quad v_3 \quad e_3 \quad v_4 \quad e_4 \quad v_5 \quad e_5 \quad v_6 \quad e_6 \quad v_7 \quad e_7 \quad v_8 \quad e_8 \quad v_9 \quad e_9 \quad u_0$
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![Diagram of a graph with vertices labeled v1, v2, v3, v4, v5, v6, and edges e1, e2, e3, e4, e5, e6, e7, e8, e9.]

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Counting grains

In Dhar criterion, each edge captures the last grain that crosses it. For any recurrent configuration \( \eta \) on \( G = (V \cup \{s\}, E) \),

\[
level(\eta) = \left( \sum_{v \in V} \eta(v) \right) + \text{deg}(s) - |E|.
\]

Let \( R_G(y) = \sum_{\eta \in \text{Rec}(G,s)} y^{level(\eta)} \)

**Theorem (López 97)**

For any graph \( G = (V \cup \{s\}, E) \),

\[
R_G(y) = \text{Tutte}_G(1, y).
\]

where \( \text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}(T)} \) counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.
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\[
level(\eta) = \left( \sum_{v \in V} \eta(v) \right) + \text{deg}(s) - |E|.
\]

Let $R_G(y) = \sum_{\eta \in \text{Rec}(G,s)} y^{level(\eta)}$

**Theorem (López 97)**

*For any graph* $G = (V \cup \{s\}, E)$,

\[
R_G(y) = \text{Tutte}_G(1, y).
\]

where $\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.
Counting grains

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where \( \text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}(T)} \) counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.
Tracking external activity while changing order on edges

With \( e_1 <_E e_2 <_E \cdots <_E e_{|E|} \) an order on the edges of \( E \), an external edge is active if it is maximal for \( <_E \) in its fundamental cycle.

Proposition

\[
\text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}<_E(T)}
\]

\( \Phi_i(\sum) \) does not depend on \( <_E \)
Tracking external activity while changing order on edges

With \( e_1 <_E e_2 <_E \cdots <_E e_{|E|} \) an order on the edges of \( E \), an external edge is *active* if it is maximal for \( <_E \) in its fundamental cycle.

\[ \text{Proposition} \]

\[ \text{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^\text{ext}_{<_E}(T) \]

does not depend on \( <_E \).

\( \{e_i, e_j\} \) is a critical pair if

- \( e_i \) is external
- \( e_j \) is on \( e_i \) fundamental cycle
- \( e_i \) and \( e_j \) are maximal on \( e_i \) fundamental cycle
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![Diagram](image)

**Proposition**

\[
T_\text{utte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\text{ext}<_E(T)}
\]
does not depend on \( <_E \)

\{\( e_i, e_j \)\} is a critical pair if

- \( e_i \) is external
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- \( e_i \) and \( e_j \) are maximal on \( e_i \) fundamental cycle

Let \( \tau_i \) exchanging \( e_i \) and \( e_{i+1} \) in \( <_E \).

\[
\Phi_i(T) = \begin{cases} 
T \Delta \{e_i, e_{i+1}\} & \text{if } \{e_i, e_{i+1}\} \text{ is a critical pair of } T \\
T & \text{otherwise}
\end{cases}
\]

**Lemma:** for all \( T \)

\[
\text{ext}_{<_E}(T) = \text{ext}_{\tau_i(<_E)}(\Phi_i(T)).
\]
Tutte Polynomial

Let a graph $G = (V, E)$ and $<_E$ an order on the edges of $E$.

$$\text{Tutte}_G(x, y) = \sum_{T \in \Sigma(G)} x^{\text{int}(T)} y^{\text{ext}(T)}$$

Active external edge: maximal in its fundamental cycle.
Active internal edge: maximal in its co-cycle.

For $G = K_4$, $\text{Tutte}_G(x, y) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y$ and $T$ weights $xy$.

When $G$ is planar, $\text{Tutte}_G(x, y) = \text{Tutte}_{G^*}(y, x)$. Then if planar and self-dual, $\text{Tutte}_G(x, y) = \text{Tutte}_G(y, x)$.
Finite graphs

- Stable configurations

- Dhar Criterion

- Bijection between recurrent and spanning trees

- Tutte polynomial

- Invariant by edge exchange

- Symmetric for self-dual planar graphs
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H. Derycke, Y. Le Borgne (LaBRI)  Square lattice, Sandpile and Recurrence  JCB 2019  10 / 27
Finite graphs

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Square lattice (biperiodicity)

- Biperiodic stable configurations
- Weak Dhar Criterion (projective sink)
- Bijection between recurrent and some spanning forests of the torus
- Tutte polynomial
- Restriction of Tutte polynomial

Invariant by edge exchange

Symmetric joint distribution of external/internal activities changing by rotation of projective sink

Symmetric for self-dual planar graphs
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Some definition of recurrence for $\mathbb{Z}^2$

From wired uniform spanning forest [Gamlin, Jarai] with an anchor burning bijection.

Local description in probability [Priezzhev, Ruelle]

Sandpile identity: $\lim_{n \to \infty} \text{dhar}^n(0^{\mathbb{Z}^2})$? [Paoletti, Caracciollo, Sportiello, Levine, Pegden, Smart...]
Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]

Convergence in terms of density [Pegden, Smart 12]
Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]

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Source: W. Pegden, $n = 2^{13}$
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Pattern in periodic zones are invariant when toppling the sink ⇒ recurrent?
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Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent?

Heuristic: locally, toppling the sink behave as the toppling of an half-plane
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2$ ($\neq (0, 0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

Direction $\vec{s}$ du puits

Diagram: Square lattice configurations before and after a toppling event.
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction \( \bar{s} \in \mathbb{Q}^2 \ (\neq (0, 0)) \) if after a forced toppling of any half-plane orthogonal to \( \bar{s} \), all other vertices in the complement topple (once).
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Direction \( \vec{s} \) du puits
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2 \setminus \{(0,0)\}$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).
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[Diagram of a square lattice with a marked direction and toppling process]

Direction \( \vec{s} \) du puits
**Definition (Weak Dhar Criterion [D., Le Borgne 2018])**

A stable configuration is recurrent for a direction $\vec{s} \in \mathbb{Q}^2 (\neq (0, 0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

![Diagram](image)
Definition (Weak Dhar Criterion \[D., \text{Le Borgne 2018}\])

A stable configuration is recurrent for a direction \(\vec{s} \in \mathbb{Q}^2 \neq (0,0)\) if after a forced toppling of any half-plane orthogonal to \(\vec{s}\), all other vertices in the complement topple (once).

\[\text{Direction } \vec{s} \text{ du puits}\]
Definition (Weak Dhar Criterion [D., Le Borgne 2018])

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Direction $\vec{s}$ du puits

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Direction $\vec{s}$ du puits
Demo
Theorem (D., Le Borgne 2018)

*The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\bar{s}$.**
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

Periodicity along the orthogonal of the sink

Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane

H. Derycke, Y. Le Borgne (LaBRI) Square lattice, Sandpile and Recurrence
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$. 

The diagram shows a sweep line method used to illustrate the theorem. The sweep line moves from left to right, updating the pattern as it goes. The numbers represent the state of the sandpile at each step, with red squares indicating sites that are updated due to the sweep line passing over them.
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$. 

\[
\begin{array}{cccccccccccc}
3 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 3 & 0 & 3 \\
1 & 3 & 0 & 3 & 3 & 1 & 3 & 0 & 3 & 3 & 1 \\
1 & 3 & 3 & 1 & 3 & 1 & 3 & 3 & 1 & 3 & 1 \\
1 & 3 & 0 & 3 & 2 & 1 & 3 & 0 & 3 & 2 & 1 \\
1 & 1 & 1 & 2 & 3 & 1 & 1 & 1 & 2 & 3 & 1 \\
\end{array}
\]
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The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

![Diagram showing the sweep line and working zone with labeling](image-url)
Theorem (D., Le Borgne 2018)

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<table>
<thead>
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<th>sweep line</th>
<th>Working zone</th>
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<tr>
<td>3 3 3 1 0 3 3 3 1 3 0 3 3 1</td>
<td>1 3 0 3 3 1 3 3 1 3 0 3 2 1</td>
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<tr>
<td>1 3 3 1 3 1 3 3 1 3 3 1</td>
<td>1 3 0 3 2 1 3 0 3 2 1</td>
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<td>1 3 0 3 2 1</td>
<td>1 1 2 3 1 1 1</td>
</tr>
<tr>
<td>1 1</td>
<td>1 1</td>
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</table>
Theorem (D., Le Borgne 2018)

*The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.*

Periodicity along the orthogonal of the sink
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\mathbf{s}$.

- Periodicity along the orthogonal of the sink
- Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane
sweep line
Spanning forests of the torus rooted on non-contractible cycles with slope $(4, -3)$.

Periodic spanning forest rooted on the half-plane.

Spanning forest of the torus with slope $(1, 0)$ incompatible with the vertical direction.

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $F_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.
Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$.

Periodic spanning forest rooted on the half-plane spanning forest of the torus with slope $(1, 0)$ incompatible with the vertical direction.

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $F_W \times H, \vec{s}$, hence excluding those of slope orthogonal to $\vec{s}$.
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Periodic spanning forest rooted on the half-plane
Spanning forests of the torus rooted on non contractible cycles with slope \((4, -3)\).

Bi-periodic spanning forest with infinite paths directed towards the sink.

Biperiodic spanning forest with infinite paths directed towards the sink.
Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$.  

Bi-periodic spanning forest with infinite paths directed towards the sink.
Spanning forests of the torus rooted on non contractible cycles with slope \((4, -3)\)

\[\begin{array}{ccccccccccccccccc}
3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 & 3 & 3 \\
0 & 1 & 3 & 3 & 0 & 1 & 3 & 3 & 0 & 1 \\
3 & 1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 \\
3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 & 3 & 3 \\
0 & 1 & 3 & 3 & 0 & 1 & 3 & 3 & 0 & 1 \\
3 & 1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 \\
3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 & 3 & 3 \\
0 & 1 & 3 & 3 & 0 & 1 & 3 & 3 & 0 & 1 \\
3 & 1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 \\
3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 & 3 & 3 \\
\end{array}\]

**Theorem [D., Le Borgne 2018]**

Recurrent configurations of period \(W \times H\) defined by weak Dhar criterion with projective sink in direction \(\vec{s}'\) are in bijections with admissible forests of \(\mathcal{F}_{W \times H, \vec{s}}\), hence excluding those of slope orthogonal to \(\vec{s}'\).
Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$.

**Theorem [D., Le Borgne 2018]**

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.
Spanning forest of the torus with slope \((1, 0)\) incompatible with the vertical direction

Spanning forests of the torus rooted on non contractible cycles with slope \((4, -3)\)

**Theorem [D., Le Borgne 2018]**

Recurrent configurations of period \(W \times H\) defined by weak Dhar criterion with projective sink in direction \(\vec{s}\) are in bijections with admissible forests of \(\mathcal{F}_{W \times H, \vec{s}}\), hence excluding those of slope orthogonal to \(\vec{s}\).
Determinantal formula [Kenyon 17] for non contractible cycle rooted spanning forests (NCRSFs)

Refinement with the infinite path’s slope

<table>
<thead>
<tr>
<th>$k \cdot j$</th>
<th>$k \cdot i$</th>
</tr>
</thead>
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<tr>
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<td></td>
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<tr>
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<tr>
<td>3</td>
<td>1528</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table:** Number of NCRSFs with $k$ cycles of slope $(i, j)$ on the torus $T_{4,4}$

Computation for $W, H \leq 9$
Inverse function

Placing the grains on the edges. • ○

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Inverse function

Placing the grains on the edges. • ○

- Orientation towards the sink

![Diagram of a triangle with labeled edges 1, 2, 3, 4, 5, 6 and arrows indicating grain directions towards the sink.]

Cycles are directed such that they are globally decreasing.

Periodicity ⇒ Maximal edge at finite distance
Inverse function

Placing the grains on the edges.

- Orientation towards the sink
- Internal: 1 grain • to the father
Inverse function

Placing the grains on the edges. • ○

▶ Orientation towards the sink
▶ Internal: 1 grain • to the father
▶ External: • depends on the position of the maximal edge on the fundamental cycle
Inverse function

Placing the grains on the edges. ● ○

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Red ray

Blue ray
Inverse function

Placing the grains on the edges. ⚫ ○

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Inverse function

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\[ e_{\text{max}} \]

\[ s \]

\[ e \]
Inverse function

Placing the grains on the edges. ⬤ ○

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Infinite decreasing path

Periodicity ⇒ Maximal edge at finite distance

H. Derycke, Y. Le Borgne (LaBRI)  Square lattice, Sandpile and Recurrence
Inverse function

Placing the grains on the edges. ▶

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- External: ○ on the other endpoint if active

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\[ s \rightarrow e \]
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Cycles are directed such that they are globally decreasing. Periodicity ⇒ Maximal edge at finite distance
Restricted Tutte Polynomial

\[ T_{W \times H, s}(x, y) = \sum_{T \in \mathcal{F}_{W \times H}} x^{\text{int}_{W \times H}(T)} y^{\text{ext}_{W \times H}(T)} \]

\( e <_s f \) if \( e \) is closer to the sink than \( f \).

Restrictions

- On NCRSF: \( \mathcal{F}_{W \times H} \).
- On the activity: on the rectangular fundamental domain \( W \times H \) consider exactly 2\( WH \) edges.
Restricted Tutte Polynomial

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\( e \prec_{s} f \) if \( e \) is closer to the sink than \( f \).

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Restricted Tutte Polynomial

External activity on $F_{3,1}$:

$$
\begin{array}{cccc}
  s & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} \\
  \vdash & 3 & 0 & 0 \\
  \vdash & 0 & 1 & 3 \\
  s & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} & \begin{array}{c}
    \begin{array}{c}
      \vdash \\
      \vdash
    \end{array}
  \end{array} \\
  \vdash & 2 & 1 & 1 & 0 \\
  \vdash & 1 & 0 & 2 & 0
\end{array}
$$

$$T_{3 \times 1, (0, 1)}(1, y) = y^3 + 3y^2 + 6y + 7$$

$$T_{3 \times 1, (-1, 0)}(1, y) = y^3 + 3y^2 + 6y + 7$$
Restricted Tutte Polynomial

External activity on $\mathcal{F}_{3,1}$:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>↑</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>←</td>
<td>0</td>
<td>1</td>
</tr>
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$\mathcal{T}_{3\times 1,(0,1)}(1, y) = y^3 + 3y^2 + 6y + 7$

$\mathcal{T}_{3\times 1,(-1,0)}(1, y) = y^3 + 3y^2 + 6y + 7$

**Theorem (D., Le Borgne 2018)**

*For any directions $s, s'$, $\mathcal{T}_{W\times H,s}(1, y) = \mathcal{T}_{W\times H,s'}(1, y)$.*
Restricted Tutte Polynomial

External activity on $F_{3,1}$:

$$\begin{array}{c|c|c|c}
  s & & & \\
  \uparrow & 3 & 0 & 0 \\
  \leftarrow & 0 & 1 & 3 \\
  \downarrow & 2 & 1 & 1 \\
  \leftarrow & 1 & 0 & 2 \\
\end{array}$$

$$\begin{array}{c|c|c|c}
  s & & & \\
  \uparrow & 3 & 0 & 0 \\
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$T_{3 \times 1,(0,1)}(1, y) = y^3 + 3y^2 + 6y + 7$

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**Theorem (D., Le Borgne 2018)**

*For any directions $s, s', T_{W \times H,s}(1, y) = T_{W \times H,s'}(1, y)$.*

Since $\mathbb{Z}^2$ is self-dual, we have:

$T_{3 \times 1,(0,1)}(x, y) = x^3y^3 + 3xy^2 + 3x^2y + 3x + 3y + 4$

$T_{3 \times 1,(-1,0)}(x, y) = x^3y^3 + 3x^2 + 3y^2 + 3xy + 3x + 3y + 1$
External activity

For each external edge $e$, there is an activity sector $[\theta_e, \theta'_e)$.

For any sector excluding all $[\theta_e)$ and $[\theta'_e)$, the external activity is invariant.

Direction of the sink
External activity

For each external edge $e_i$, there is an activity sector $\left[ \theta_{e_i}, \theta'_{e_i} \right)$. For any sector excluding all $\left( \theta_{e_i} \right)$ and $\left( \theta'_{e_i} \right)$, the external activity is invariant.

Direction of the sink

- Convex hulls of fundamental cycles.
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- Active $\Rightarrow$ Convex hull corner
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Convex hulls of fundamental cycles.
Active $\Rightarrow$ Convex hull corner
Critical pair exchange: Rotation step

$\rightarrow s = (1, 0)$
Critical pair exchange: Rotation step

$s = (1, 1)$
Critical pair exchange: Rotation step

\[ s = (1, 1) \]
Critical pair exchange: Rotation step

\[ s = (0, 1) \]
Critical pair exchange: Rotation step

\[ s = (0, 1) \]
Critical pair exchange: changing forest slope
<table>
<thead>
<tr>
<th>Finite graphs</th>
<th>Square lattice (biperiodicity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Stable configurations</td>
<td>▶ Biperiodic stable configurations</td>
</tr>
<tr>
<td>▶ Dhar Criterion</td>
<td>▶ Weak Dhar Criterion (projective sink)</td>
</tr>
<tr>
<td>▶ Bijection between recurrent and spanning trees</td>
<td>▶ Bijection <em>recurrent</em> and some spanning forests of the torus</td>
</tr>
<tr>
<td>▶ Tutte polynomial</td>
<td>▶ Restriction of Tutte polynomial</td>
</tr>
<tr>
<td>▶ Invariant by edge exchange</td>
<td>▶ Distribution of external activity invariant by rotation of projective sink</td>
</tr>
<tr>
<td>▶ Symmetric for self-dual planar graphs</td>
<td>▶ Symmetric joint distribution of external/internal activities <em>changing</em> by rotation</td>
</tr>
</tbody>
</table>
Conclusion

We have
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- Weak Dhar Criterion efficient for biperiodic configurations
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- Bijection with NCRSFs, extending the definition of biperiodic recurrent configurations
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- Involution on NCRSFs for atomic rotation preserving this distribution
Conclusion

Perspectives
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- $T_{W \times H, s}(x, y)$ depends on $s$
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- $\mathcal{T}_{W \times H, s}(x, y)$ depends on $s$
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- $T_{W \times H, s}(x, y)$ depends on $s$
- Iteration of the rotation step can take several rounds before the identity
- What about other orders?
  - Experiments: periodic decreasing orders towards a direction is enough:
    
    $e <_E f \Rightarrow e + (iW, jH) <_E f + (iW, jH)$ and
    
    $\langle s, (iW, jH) \rangle > 0 \Rightarrow e + (iW, jH) <_E e$
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    - $\langle s, (iW, jH) \rangle > 0 \Rightarrow e + (iW, jH) <_E e$
  - Only decreasing, or only periodic
  - Anything else
THANK YOU
Markov Chain for $G = (V \cup \{S\}, E)$

- States: stable configurations on $G$
- Transition: Add a particle to a vertex chosen uniformly and stabilize

The recurrent states are called recurrent configurations.
The stationary distribution is uniform on the recurrent configurations.

**Dhar Criterion** A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)
Markov Chain for \( G = (V \cup \{S\}, E) \)

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**Dhar Criterion** A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)
Close to [Pegden and Smart, 2017]

Figure: Each non blue zone is described by a quadratic form. [arxiv:1708.09432]
Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

\[ M(a, b, c) = \begin{pmatrix} c + a & b \\ b & c - a \end{pmatrix} \]

The number of topples is:

\[ h(x) = \left\lceil \frac{1}{2} x^t M(a, b, c) x \right\rceil \]

\[ = (c + a)x^2 + 2bxy + (c - a)y^2 \]

Sample for 
\[ M(0.25, 0.875, 2.125) \]
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- It’s periodic for \( a, b, c \in \mathbb{Q} \)
- But it may be negative and/or unstable!

It may be stabilized without changing density of grains.
A definition of recurrence for periodic stable configurations

Pattern + two dimensional period \((\vec{p}_1, \vec{p}_2)\).

\[ \forall x \in \mathbb{Z}^2 u(x) = u(x + \vec{p}_1) = u(x + \vec{p}_2) \]

\[
\begin{array}{cccc}
3 & 1 & 1 & \quad 3 & 3 & 1 & 1 & \quad 3 & 3 & 1 \\
3 & 3 & 3 & \quad 0 & 3 & 3 & 3 & \quad 0 & 3 & 3 \\
0 & 1 & 3 & \quad \vec{p}_2 & 3 & 0 & 1 & 3 & \quad 3 & 0 & 1 \\
3 & 1 & 1 & \quad 3 & 3 & 1 & 1 & \quad 3 & 3 & 1 \\
3 & 3 & 3 & \quad 0 & 3 & 3 & 3 & \quad 0 & 3 & 3 \\
0 & 1 & 3 & \quad 3 & 0 & 1 & 3 & \quad 3 & 0 & 1 \\
3 & 1 & 1 & \quad 3 & 3 & \vec{p}_1 & 1 & 1 & \quad 3 & 3 & 1 \\
3 & 3 & 3 & \quad 0 & 3 & 3 & 3 & \quad 0 & 3 & 3 \\
3 & 3 & 3 & \quad 0 & 3 & 3 & 3 & \quad 0 & 3 & 3 \\
\end{array}
\]
Periodicity on $-s$

**Lemme**

If a periodic configuration is recurrent, then there exists a position $y = t_1$ for which all vertices of the first period are toppled.

We have $\text{Period 1} \subset E_{0, t_1} \Rightarrow E_{0, t_1} = E_{H, t_1 - H}$ and $v \in E_{0, t_1} \Rightarrow v + H\vec{y} \in E_{H, t_1}$. Then $E_{H, t_1} \supset \text{Period 2}$. 

H. Derycke, Y. Le Borgne (LaBRI) Square lattice, Sandpile and Recurrence