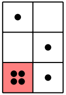
Une notion de récurrence dans le modèle du tas de sable sur le réseau carré

Henri Derycke
joint work with Yvan Le Borgne

LaBRI

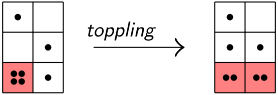
JCB 2019, Février 11-13, Bordeaux

Sandpile Model [Bak, Tang, Wiesenfeld 87]



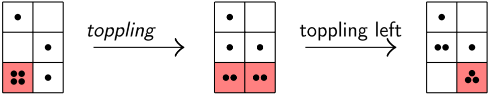
Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



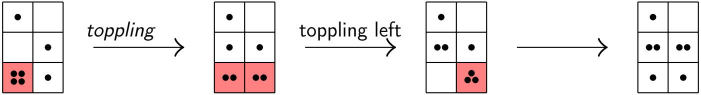
Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling $u : \eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



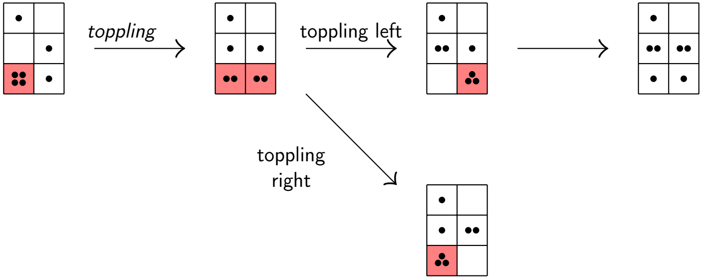
Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling $u : \eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling $u : \eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]

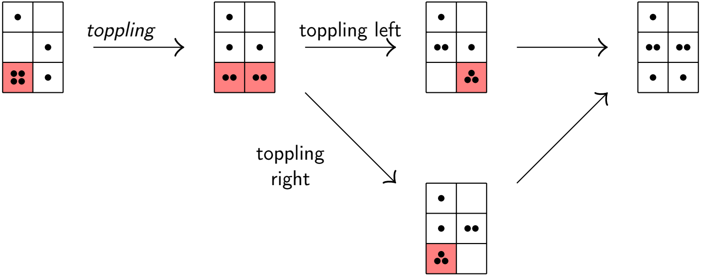


Configuration: $\eta : V \mapsto \mathbb{N}$

$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.

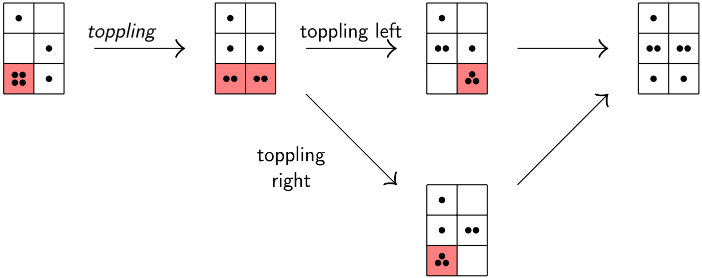
Toppling u : $\eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



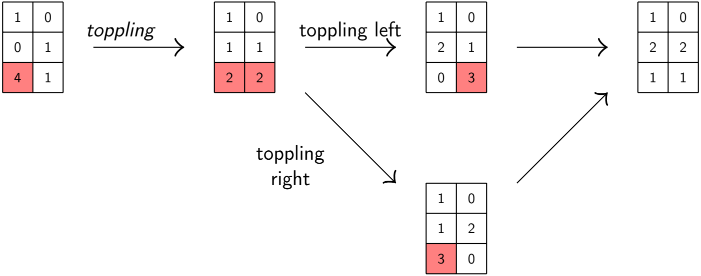
Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling $u : \eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



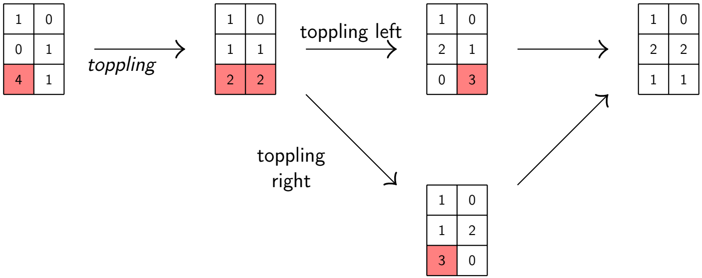
Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling u : $\eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.
The order of toppling does not change the result: $\eta \rightarrow \eta + \sum_{v \in V} a_v \Delta^{(v)}$.

Sandpile Model [Bak, Tang, Wiesenfeld 87]



Configuration: $\eta : V \mapsto \mathbb{N}$
$v \in V$ is *unstable* for η if $\eta(v) \geq \deg(v)$, it is *stable* otherwise.
Toppling u : $\eta \mapsto \eta + \Delta^{(u)}$ If u is unstable, the toppling is legal. It is forced otherwise.
The order of toppling does not change the result: $\eta \to \eta + \sum_{v \in V} a_v \Delta^{(v)}$.

Stabilisation



Stabilisation : while a vertex is unstable, topple it.

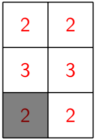
The sink

How to stabilize (even with a large number of grains)?

2	2
3	3
2	2

The sink

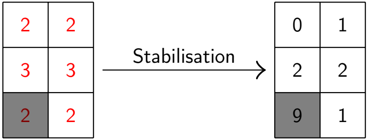
How to stabilize (even with a large number of grains)?



We distinguish a vertex as the sink that won't topple.

The sink

How to stabilize (even with a large number of grains)?

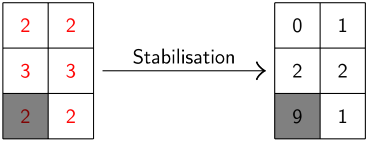


We distinguish a vertex as the sink that won't topple.

The sink guarantees that the stabilisation of any configuration η terminates and we note the result *mathsf stab*(η).

The sink

How to stabilize (even with a large number of grains)?



We distinguish a vertex as the sink that won't topple.

The sink guarantees that the stabilisation of any configuration η terminates and we note the result $mathsf\, stab(\eta)$.

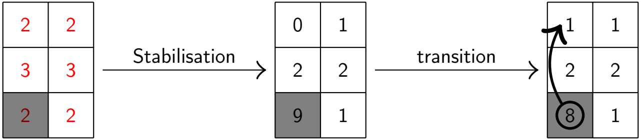
Markov Chain

- States: stable configurations on G
- Transition: Add a particle from the sink to a vertex chosen uniformly and stabilize

Recurrent states are in the same connected component.

The sink

How to stabilize (even with a large number of grains)?



We distinguish a vertex as the sink that won't topple.

The sink guarantees that the stabilisation of any configuration η terminates and we note the result $mathsf\, stab(\eta)$.

Markov Chain

- States: stable configurations on G
- Transition: Add a particle from the sink to a vertex chosen uniformly and stabilize

Recurrent states are in the same connected component.

Test of recurrence

Dhar operator

Topple the sink (forced), then stabilize: $\mathsf{dhar}(\eta) := \mathsf{stab}(\eta + \Delta^{(s)})$

Test of recurrence

Dhar operator

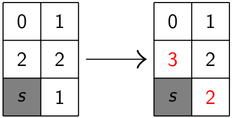
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$

0	1
2	2
s	1

Test of recurrence

Dhar operator

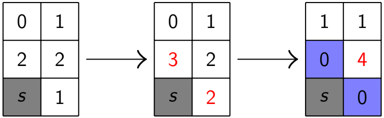
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Test of recurrence

Dhar operator

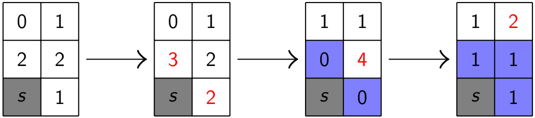
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Test of recurrence

Dhar operator

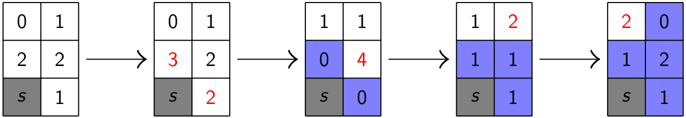
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Test of recurrence

Dhar operator

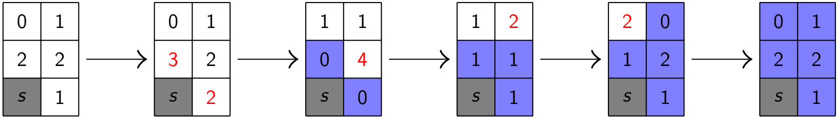
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Test of recurrence

Dhar operator

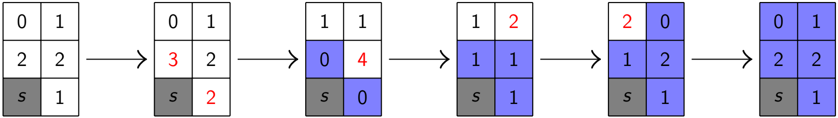
Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Test of recurrence

Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



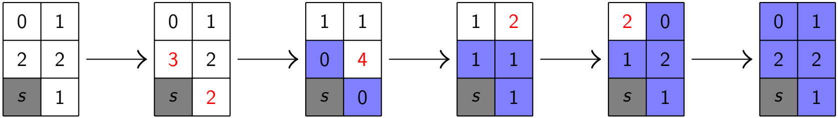
Dhar Criterion

A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.

Test of recurrence

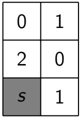
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

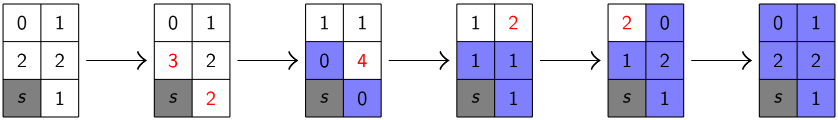
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

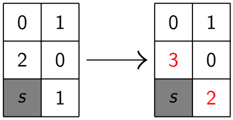
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

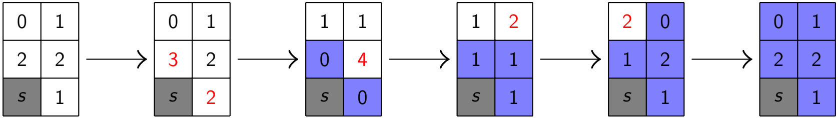
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

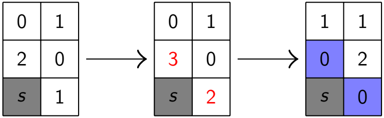
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

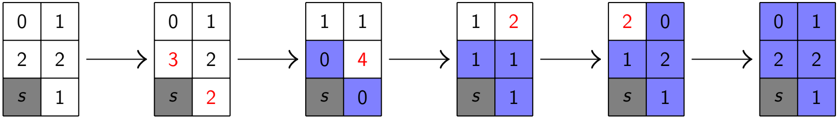
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

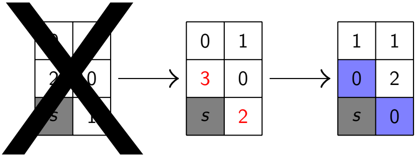
Dhar operator

Topple the sink (forced), then stabilize: $\text{dhar}(\eta) := \text{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

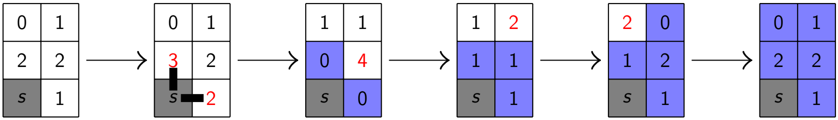
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

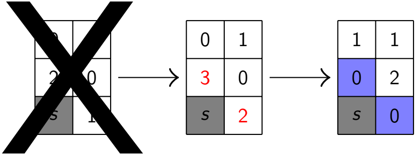
Dhar operator

Topple the sink (forced), then stabilize: $\text{dhar}(\eta) := \text{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

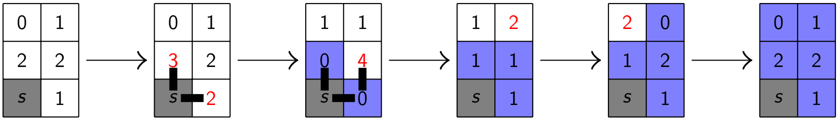
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

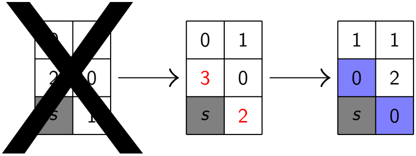
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

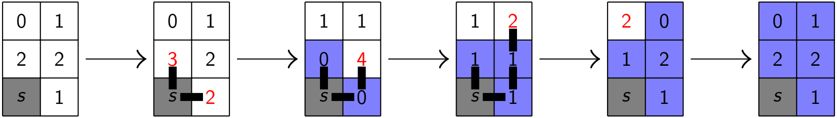
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

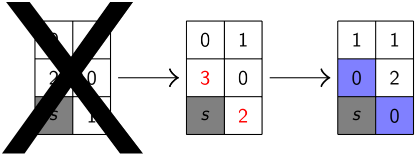
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

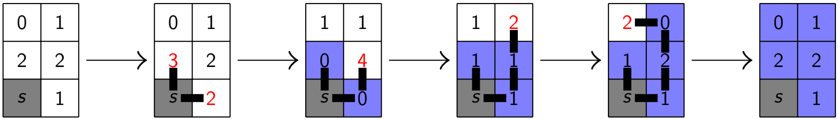
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

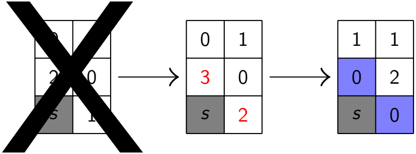
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

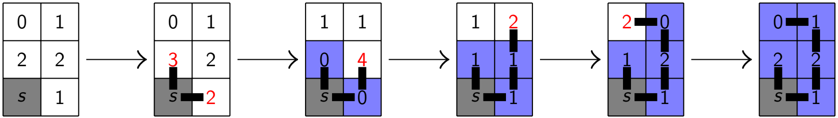
A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.



Test of recurrence

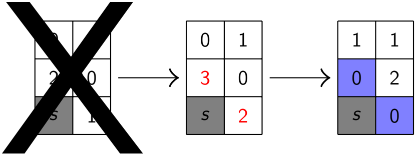
Dhar operator

Topple the sink (forced), then stabilize: $\mathrm{dhar}(\eta) := \mathrm{stab}(\eta + \Delta^{(s)})$



Dhar Criterion

A stable configuration is recurrent iff it is a fixed point of the Dhar operator. Then, each vertex topples exactly once while the operator execution.

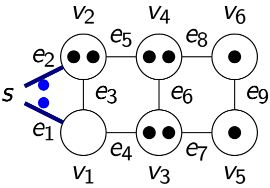


Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: s,

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

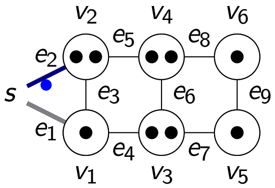
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

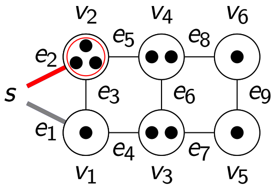
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

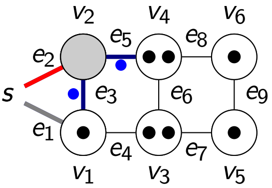
Edge-vertex traversal: $s, e_1, e_2,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

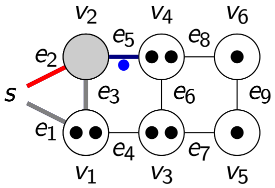
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

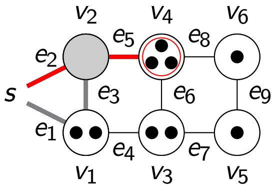
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

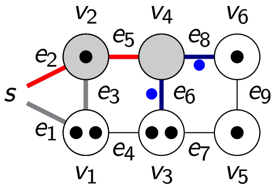
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

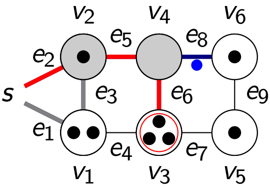
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

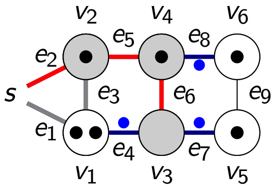
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

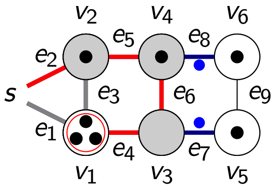
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

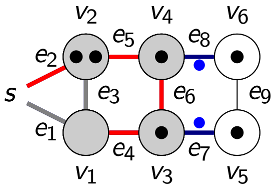
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

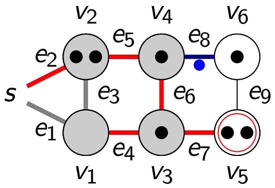
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

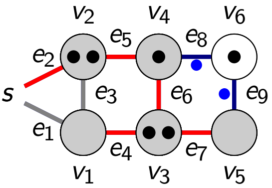
The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:

Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)

Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)

Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

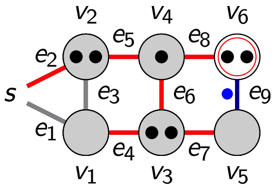
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

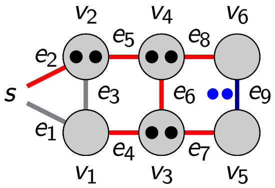
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5, e_8,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

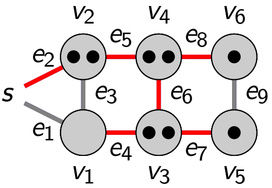
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5, e_8, v_6,$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

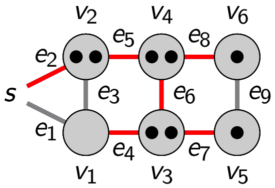
Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5, e_8, v_6, e_9$

Bijections with spanning trees

Theorem (Dhar, Majumdar 92)

The recurrent configurations for a finite graph G and its spanning trees are in bijection.

Several bijections:
Dhar/Majumdar 92 (e.g. Haglund bounce's path for sorted recurrents on K_n)
Bernardi 06 (e.g. Visiting frontier in planar maps between primal/dual spanning trees)
Cori/Le Borgne 03 (CLB) (e.g. Dhar criterion more uniform in space)




```
Mark edges incident to the sink as pending edges.
While there is a pending edge
  Get the closest pending edge to the sink
  Process the grain(s) on the edge
  If a vertex become unstable, topple it and mark its
    untreated incident edges as pending edges.
```

Edge-vertex traversal: $s, e_1, e_2, v_2, e_3, e_5, v_4, e_6, v_3, e_4, v_1, e_7, v_5, e_8, v_6, e_9$

Counting grains

In Dhar criterion, each edge captures the last grain that crosses it.
For any recurrent configuration η on $G = (V \cup \{s\}, E)$,

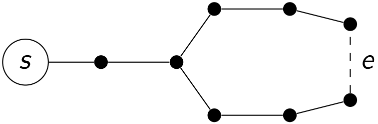
$$level(\eta) = \left(\sum_{v \in V} \eta(v) \right) + \deg(s) - |E|.$$

Let $R_G(y) = \sum_{\eta \in \mathrm{Rec}(G,s)} y^{level(\eta)}$

Theorem (López 97)

For any graph $G = (V \cup \{s\}, E)$,

$$R_G(y) = \mathrm{Tutte}_G(1, y).$$



where $\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.

Counting grains

In Dhar criterion, each edge captures the last grain that crosses it.
For any recurrent configuration η on $G = (V \cup \{s\}, E)$,

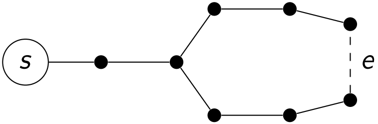
$$level(\eta) = \left(\sum_{v \in V} \eta(v)\right) + \deg(s) - |E|.$$

Let $R_G(y) = \sum_{\eta \in \mathrm{Rec}(G,s)} y^{level(\eta)}$

Theorem (López 97)

For any graph $G = (V \cup \{s\}, E)$,

$$R_G(y) = \mathrm{Tutte}_G(1, y).$$



where $\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.

Counting grains

In Dhar criterion, each edge captures the last grain that crosses it.
For any recurrent configuration η on $G = (V \cup \{s\}, E)$,

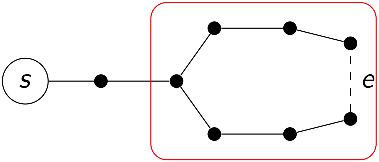
$$level(\eta) = \left(\sum_{v \in V} \eta(v)\right) + \deg(s) - |E|.$$

Let $R_G(y) = \sum_{\eta \in \mathrm{Rec}(G,s)} y^{level(\eta)}$

Theorem (López 97)

For any graph $G = (V \cup \{s\}, E)$,

$$R_G(y) = \mathrm{Tutte}_G(1, y).$$



where $\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.

Counting grains

In Dhar criterion, each edge captures the last grain that crosses it.
For any recurrent configuration η on $G = (V \cup \{s\}, E)$,

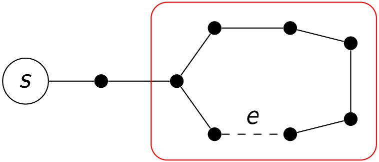
$$level(\eta) = \left(\sum_{v \in V} \eta(v) \right) + \deg(s) - |E|.$$

Let $R_G(y) = \sum_{\eta \in \mathrm{Rec}(G,s)} y^{level(\eta)}$

Theorem (López 97)

For any graph $G = (V \cup \{s\}, E)$,

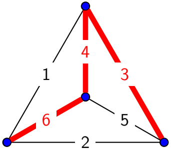
$$R_G(y) = \mathrm{Tutte}_G(1, y).$$



where $\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}(T)}$ counts on spanning trees the number of active external edges: external edges that are maximal in their fundamental cycle.

Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \cdots <_E e_{|E|}$ an order on the edges of E, an external edge is *active* if it is maximal for $<_E$ in its fundamental cycle.

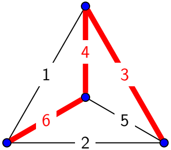


Proposition

$\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}_{<_E}(T)}$ does not depend on $<_E$

Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \cdots <_E e_{|E|}$ an order on the edges of E, an external edge is *active* if it is maximal for $<_E$ in its fundamental cycle.



Proposition

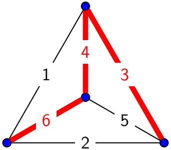
$\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}_{<_E}(T)}$ does not depend on $<_E$

$\{e_i, e_j\}$ is a critical pair if

- e_i is external
- e_j is on e_i fundamental cycle
- e_i and e_j are maximal on e_i fundamental cycle

Tracking external activity while changing order on edges

With $e_1 <_E e_2 <_E \cdots <_E e_{|E|}$ an order on the edges of E, an external edge is *active* if it is maximal for $<_E$ in its fundamental cycle.



Proposition

$\mathrm{Tutte}_G(1, y) = \sum_{T \in \Sigma(G)} y^{\mathrm{ext}_{<_E}(T)}$ does not depend on $<_E$

$\{e_i, e_j\}$ is a critical pair if

- e_i is external
- e_j is on e_i fundamental cycle
- e_i and e_j are maximal on e_i fundamental cycle

Let τ_i exchanging e_i and e_{i+1} in $<_E$.

$$\Phi_i(T) = \begin{cases} T\Delta\{e_i, e_{i+1}\} & \text{if } \{e_i, e_{i+1}\} \text{ is a critical pair of } T \\ T & \text{otherwise} \end{cases}$$

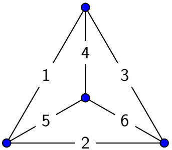
Lemma: for all T $\mathrm{ext}_{<_E}(T) = \mathrm{ext}_{\tau_i(<_E)}(\Phi_i(T))$.

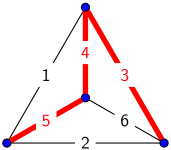
Tutte Polynomial

Let a graph $G = (V, E)$ and $<_E$ an order on the edges of E.

$$\mathrm{Tutte}_G(x, y) = \sum_{T \in \Sigma(G)} x^{\mathrm{int}(T)} y^{\mathrm{ext}(T)}$$

Active external edge: maximal in its fundamental cycle.
Active internal edge: maximal in its co-cycle.





e_6 is active with fundamental cycle (e_3, e_4, e_6). e_5 is active with co-cycle (e_1, e_2, e_5).

For $G = K_4$, $\mathrm{Tutte}_G(x, y) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y$ and T weights xy.
When G is planar, $\mathrm{Tutte}_G(x, y) = \mathrm{Tutte}_{G^*}(y, x)$. Then if planar and self-dual, $\mathrm{Tutte}_G(x, y) = \mathrm{Tutte}_G(y, x)$

Checkpoint

Finite graphs
▷ Stable configurations
▷ Dhar Criterion
▷ Bijection between recurrent and spanning trees
▷ Tutte polynomial
▷ Invariant by edge exchange
▷ Symmetric for self-dual planar graphs

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	
▷ Dhar Criterion	
▷ Bijection between recurrent and spanning trees	
▷ Tutte polynomial	
▷ Invariant by edge exchange	
▷ Symmetric for self-dual planar graphs	

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	
▷ Bijection between recurrent and spanning trees	
▷ Tutte polynomial	
▷ Invariant by edge exchange	
▷ Symmetric for self-dual planar graphs	

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	
▷ Tutte polynomial	
▷ Invariant by edge exchange	
▷ Symmetric for self-dual planar graphs	

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection *recurrent* and some spanning forests of the torus
▷ Tutte polynomial	
▷ Invariant by edge exchange	
▷ Symmetric for self-dual planar graphs	

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection *recurrent* and some spanning forests of the torus
▷ Tutte polynomial	▷ Restriction of Tutte polynomial
▷ Invariant by edge exchange	
▷ Symmetric for self-dual planar graphs	

Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection *recurrent* and some spanning forests of the torus
▷ Tutte polynomial	▷ Restriction of Tutte polynomial
▷ Invariant by edge exchange	▷ Distribution of external activity invariant by rotation of projective sink

▷ Symmetric for self-dual planar graphs

Checkpoint

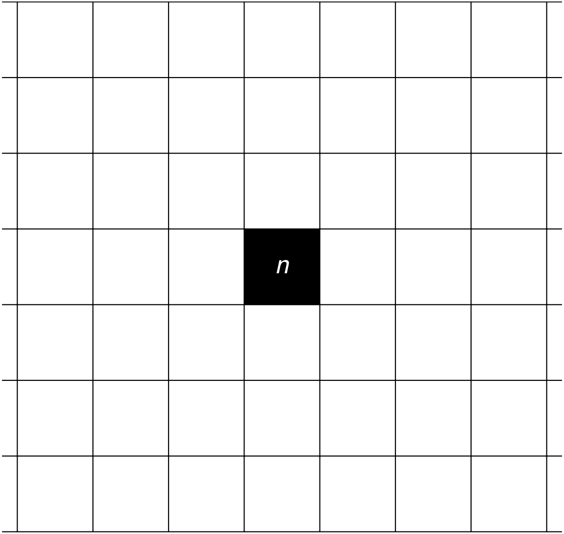
Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection *recurrent* and some spanning forests of the torus
▷ Tutte polynomial	▷ Restriction of Tutte polynomial
▷ Invariant by edge exchange	▷ Distribution of external activity invariant by rotation of projective sink
▷ Symmetric for self-dual planar graphs	▷ Symmetric joint distribution of external/internal activities *changing* by rotation

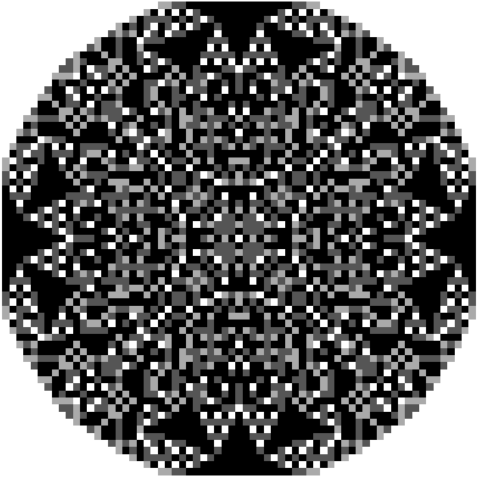
Some definition of recurrence for $\mathbb{Z}^2$

From wired uniform spanning forest [Gamlin, Jarai] with an anchor burning bijection.

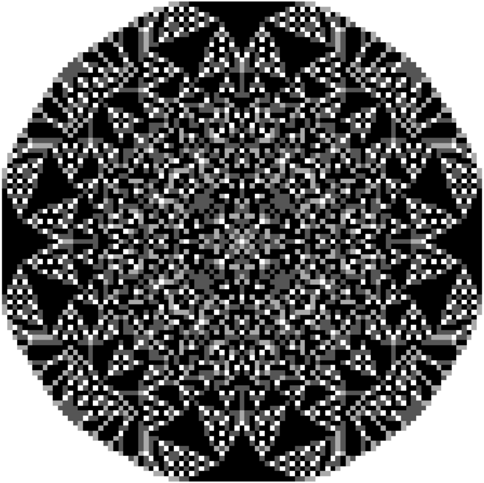
Local description in probability [Priezzhev, Ruelle]

Sandpile identity: $\lim_{n \to \infty} \mathrm{dhar}^n(0^{\mathbb{Z}^2})$? [Paoletti, Caracciollo, Sportiello, Levine, Pegden, Smart...]

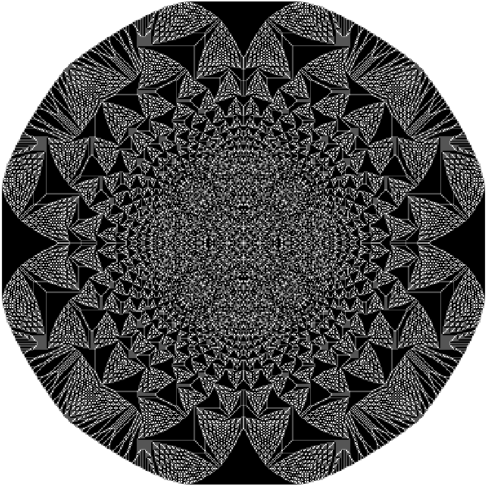

n



Source: W.Pegden, $n = 2^{13}$

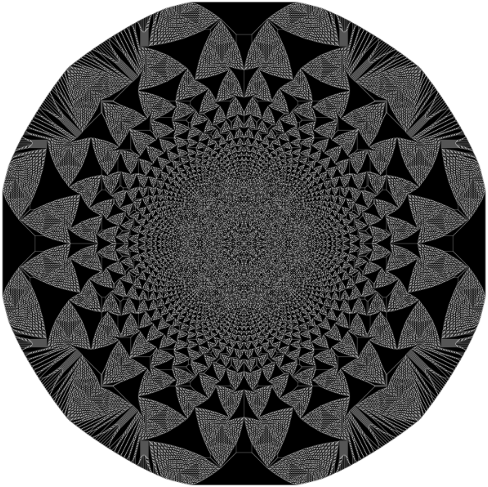


Source: W.Pegden, $n = 2^{14}$



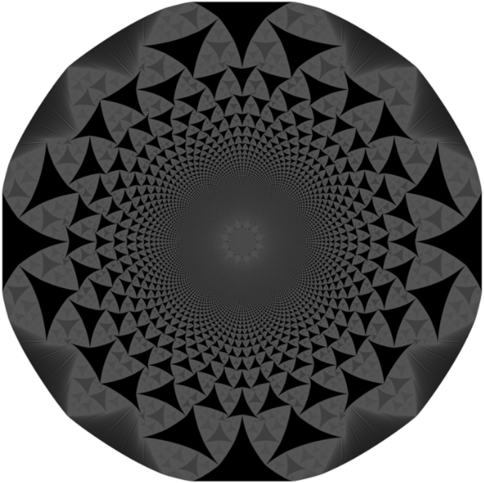
Source: W.Pegden, $n = 2^{18}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]



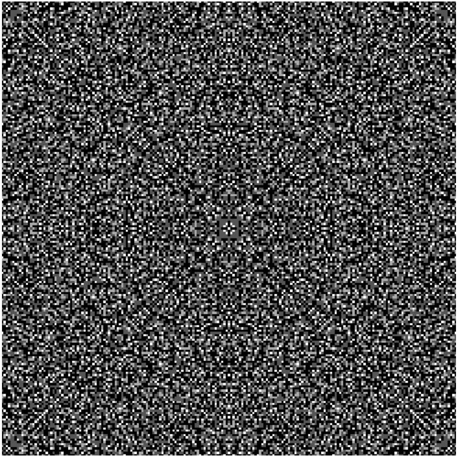
Source: W.Pegden, $n = 2^{20}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]



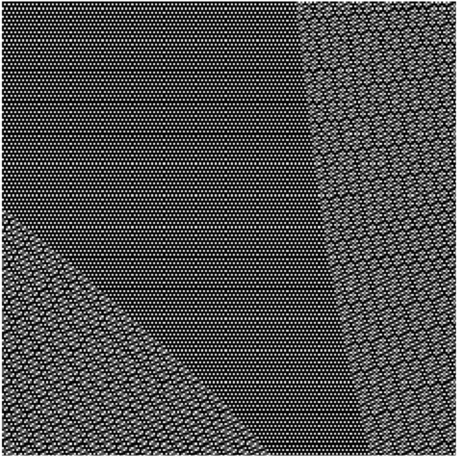
Source: W.Pegden, $n = 2^{30}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
- Convergence in terms of density [Pegden, Smart 12]



Source: W.Pegden, $n = 2^{30}$

- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
- Convergence in terms of density [Pegden, Smart 12]

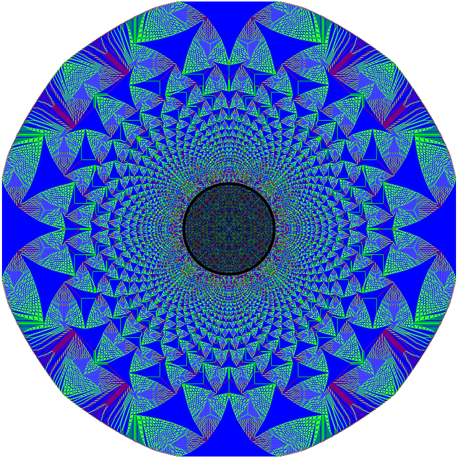


Source: W.Pegden, $n = 2^{30}$

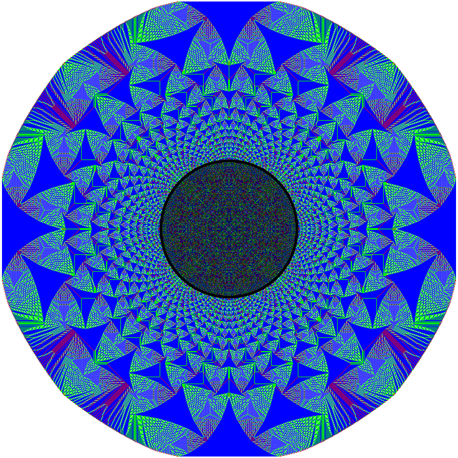
- Fractal structure [Creutz, Bak, Tang 90, Ostojic 03, Dhar Sadhu 08]
- Convergence in terms of density [Pegden, Smart 12]

Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?

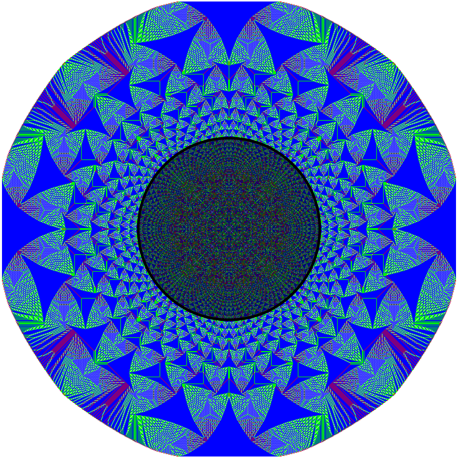
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



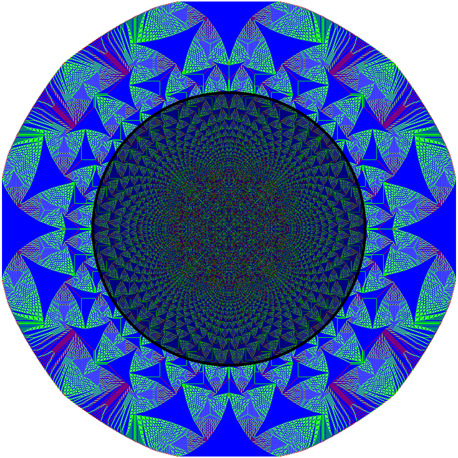
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



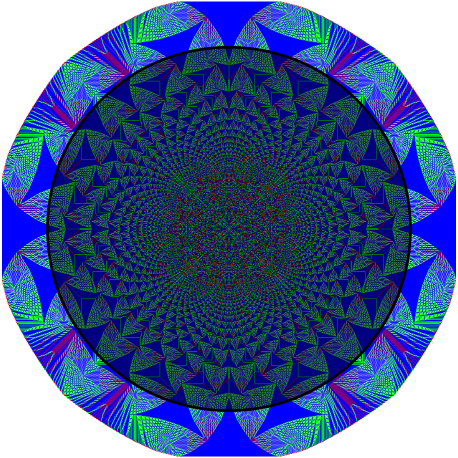
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



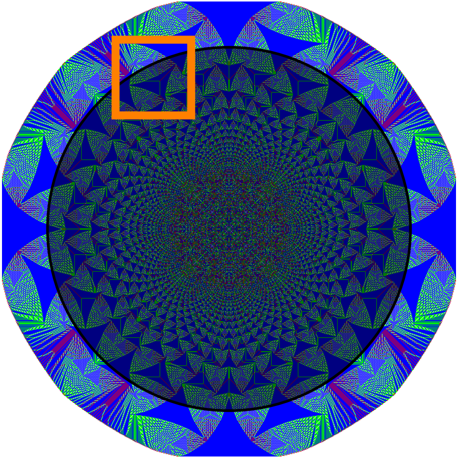
Pattern in periodic zones are invariant when toppling the sink ⇒ recurrent ?



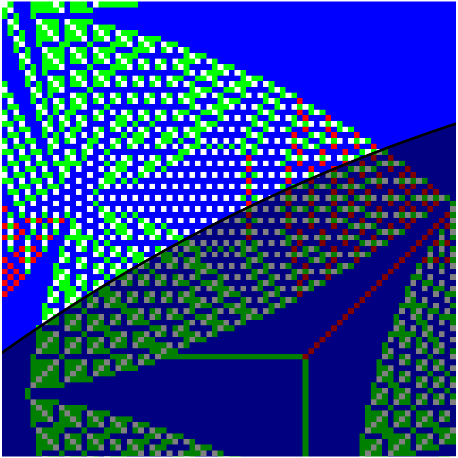
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



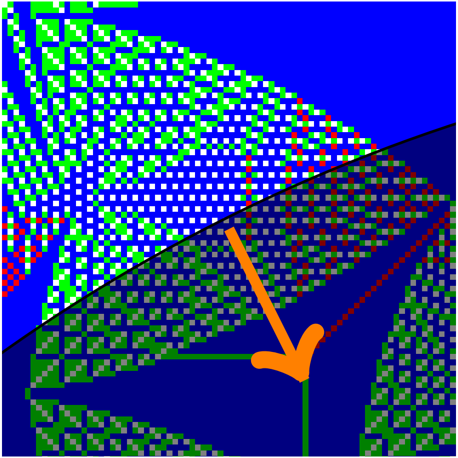
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



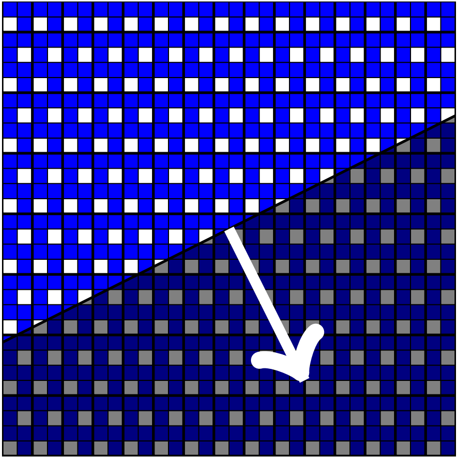
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?



Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?

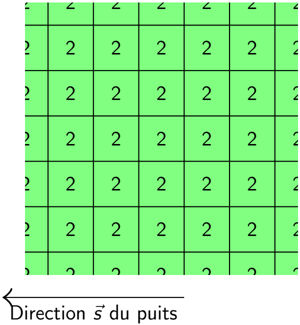


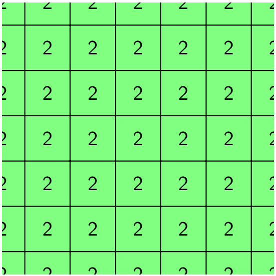
Pattern in periodic zones are invariant when toppling the sink $\Rightarrow$ recurrent ?

Heuristic: locally, toppling the sink behave as the toppling of an half-plane

Definition (Weak Dhar Criterion [D., Le Borgne 2018])

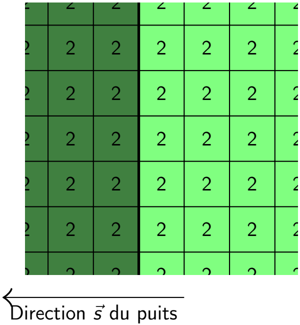
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

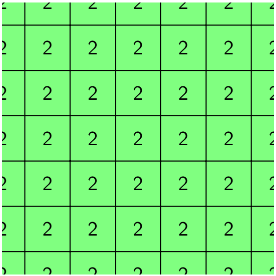




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

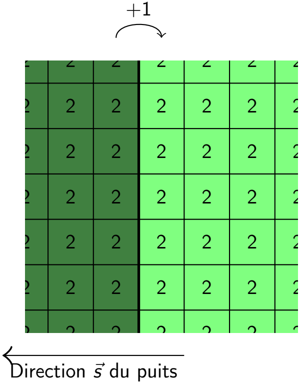
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

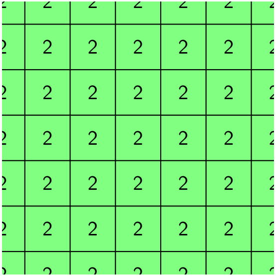




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

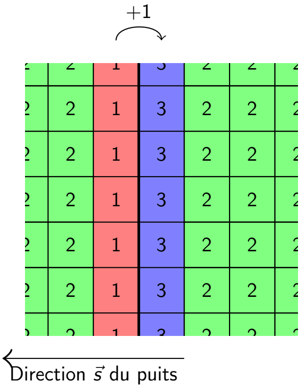
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

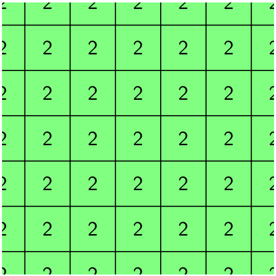




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

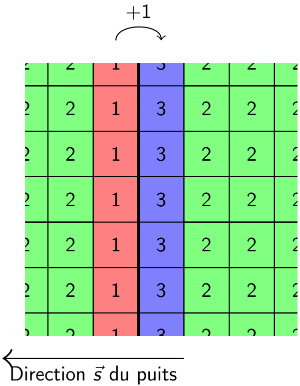
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

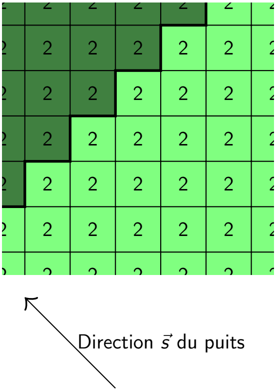




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

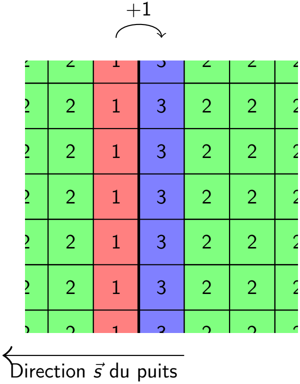
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

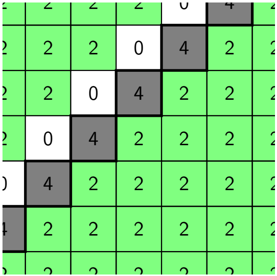




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

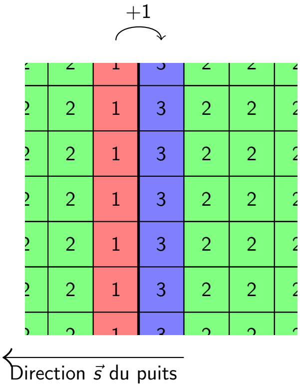
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

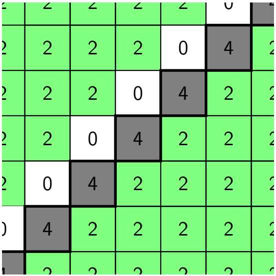




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

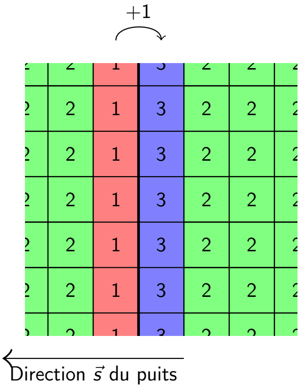
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

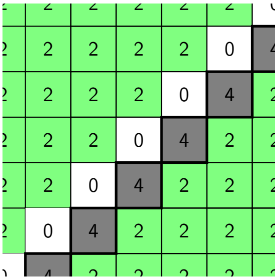




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

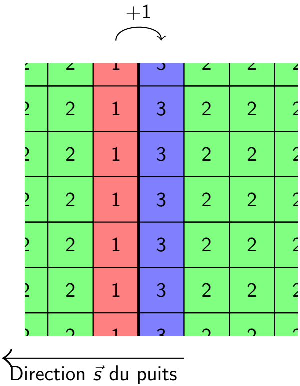
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

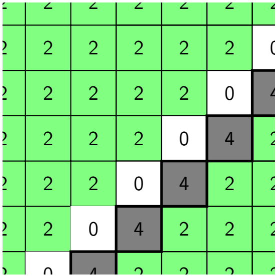




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

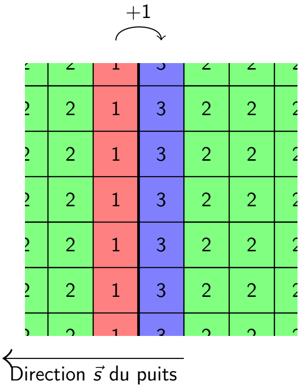
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

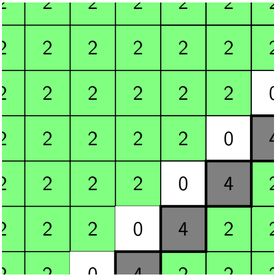




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

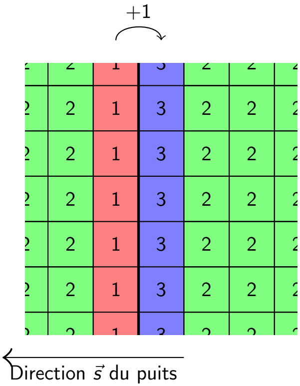
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

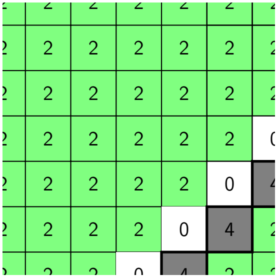




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

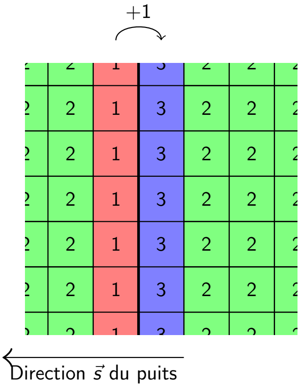
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

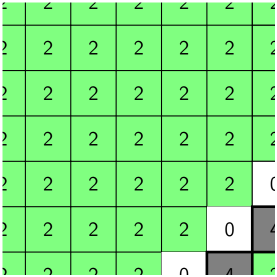




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

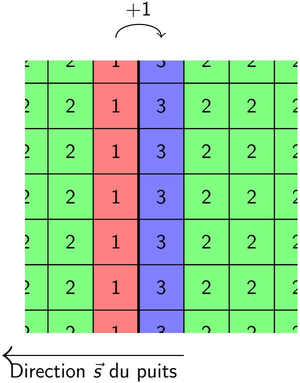
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

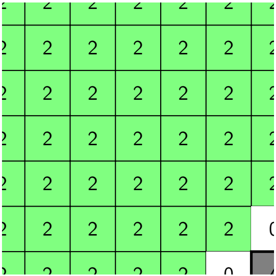




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

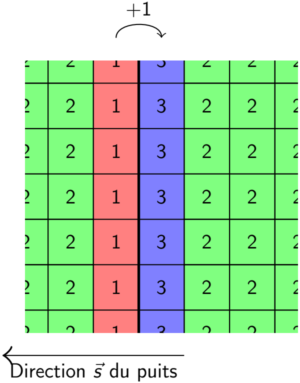
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

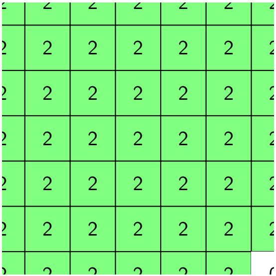




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

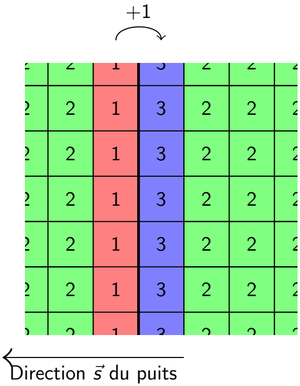
A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ ($\neq (0,0)$) if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

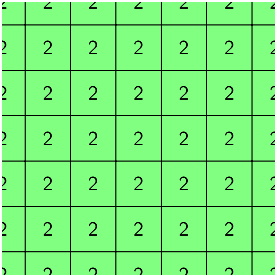




Definition (Weak Dhar Criterion [D., Le Borgne 2018])

A stable configuration is recurrent for a direction $\vec{s} \in \mathbf{Q}^2$ $(\neq (0,0))$ if after a forced toppling of any half-plane orthogonal to $\vec{s}$, all other vertices in the complement topple (once).

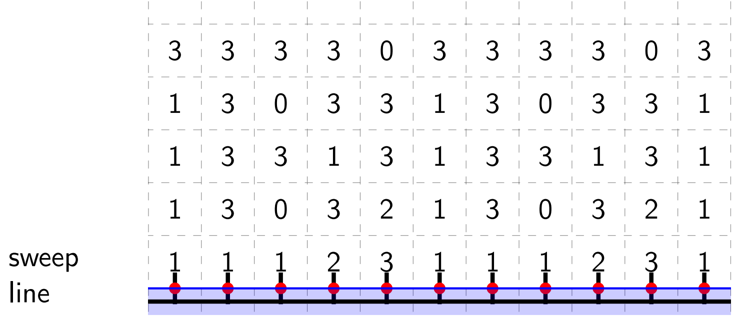




Demo

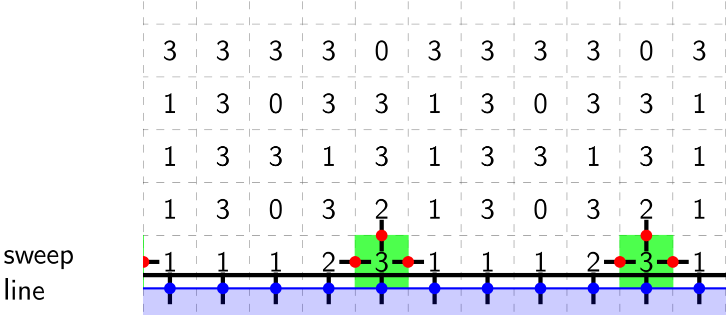
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



Theorem (D., Le Borgne 2018)

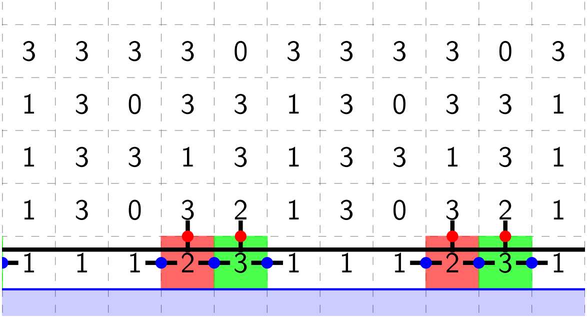
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



Theorem (D., Le Borgne 2018)

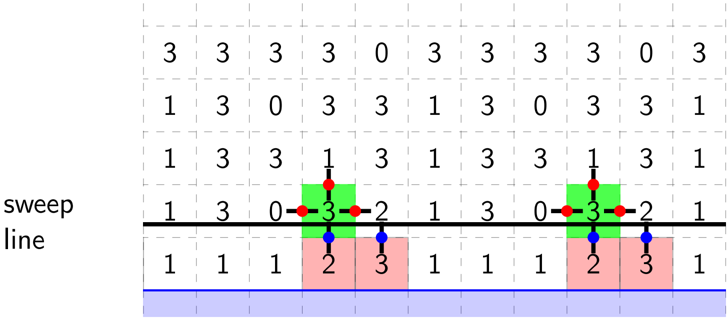
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

sweep line



Theorem (D., Le Borgne 2018)

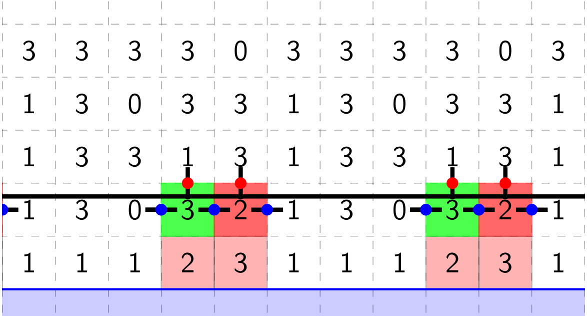
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



Theorem (D., Le Borgne 2018)

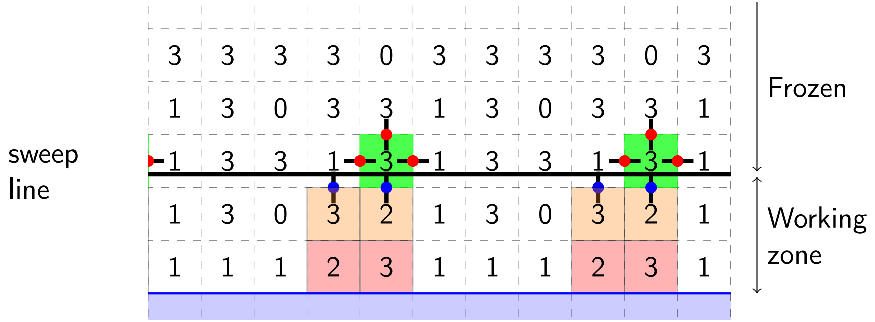
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.

sweep line



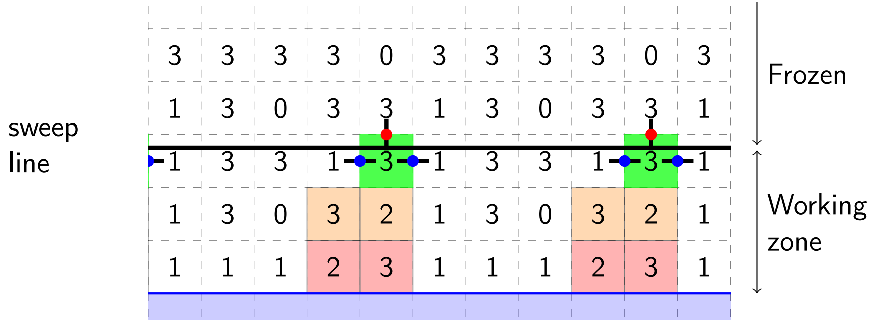
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



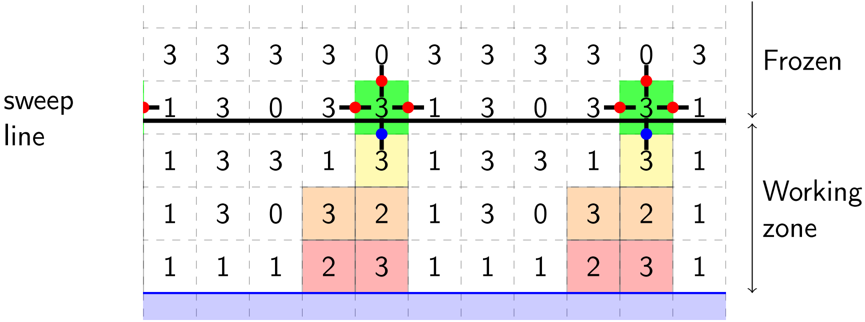
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



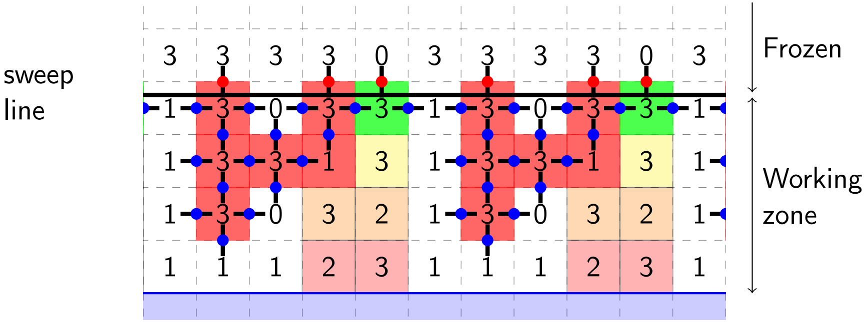
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



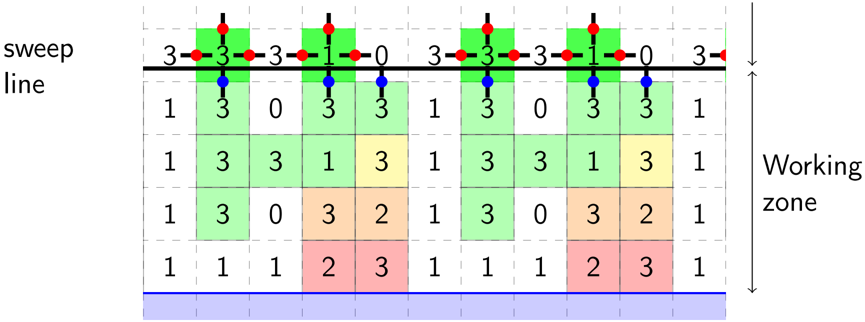
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



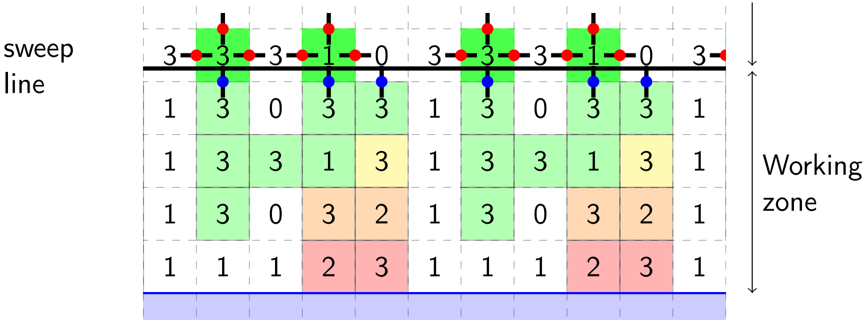
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



Theorem (D., Le Borgne 2018)

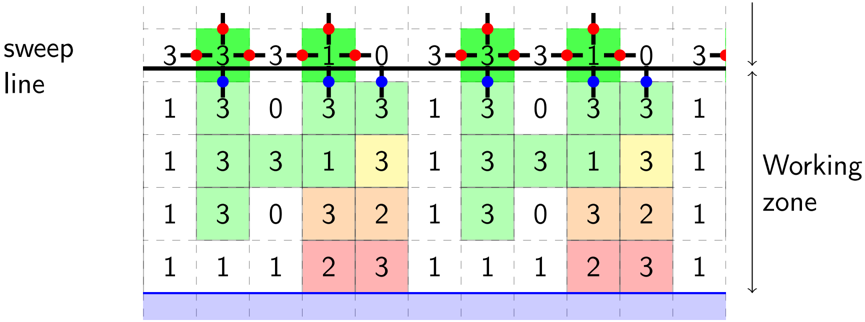
The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



- Periodicity along the orthogonal of the sink

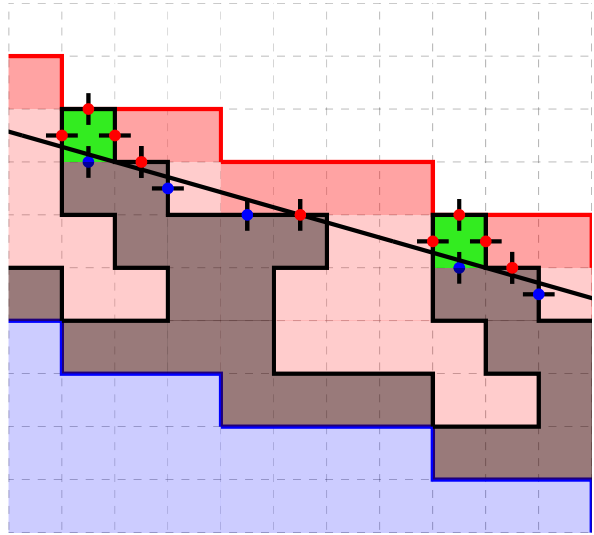
Theorem (D., Le Borgne 2018)

The Weak Dhar Criterion is decidable with in time bounded by a function of the dimension of the pattern and the direction $\vec{s}$.



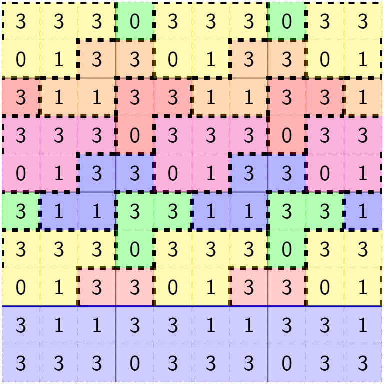
- Periodicity along the orthogonal of the sink
- Ultimately periodicity in the opposite direction of the sink, whatever the starting half-plane

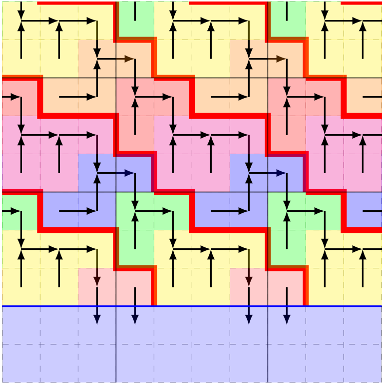
sweep line



3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3

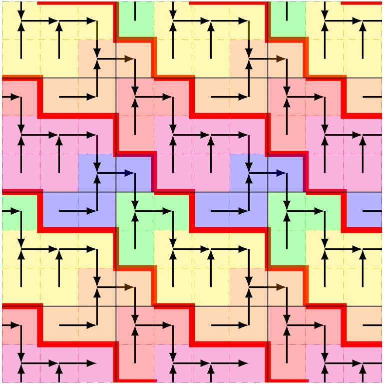
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3



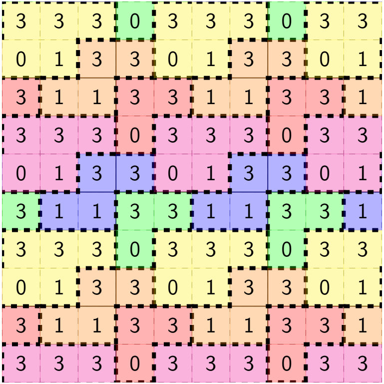


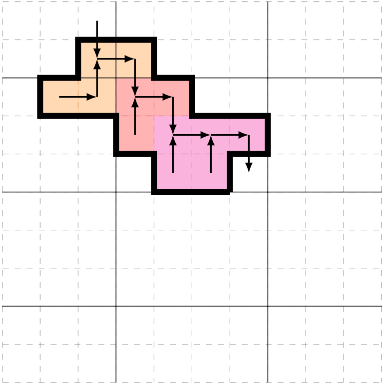
Periodic spanning forest rooted on the half-plane

3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3
0	1	3	3	0	1	3	3	0	1
3	1	1	3	3	1	1	3	3	1
3	3	3	0	3	3	3	0	3	3

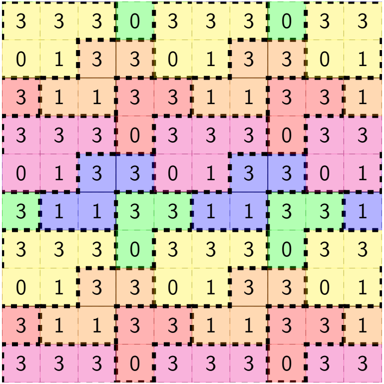


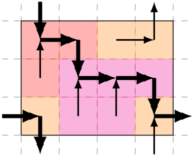
Biperiodic spanning forest with infinite paths directed towards the sink





Biperiodic spanning forest with infinite paths directed towards the sink

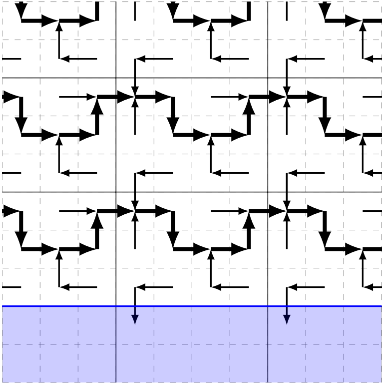


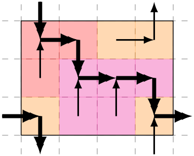


Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.

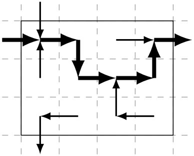




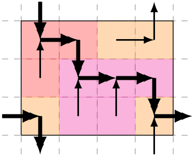
Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W \times H, \vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.



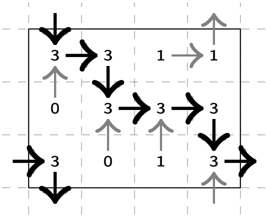
Spanning forest of the torus with slope $(1, 0)$ incompatible with the vertical direction



Spanning forests of the torus rooted on non contractible cycles with slope $(4, -3)$

Theorem [D., Le Borgne 2018]

Recurrent configurations of period $W \times H$ defined by weak Dhar criterion with projective sink in direction $\vec{s}$ are in bijections with admissible forests of $\mathcal{F}_{W\times H,\vec{s}}$, hence excluding those of slope orthogonal to $\vec{s}$.



Determinantal formula [Kenyon 17] for non contractible cycle rooted spanning forests (NCRSFs)

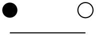
Refinement with the infinite path's slope

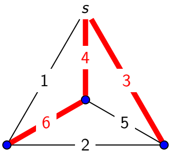
$k \cdot j$	$k \cdot i$: 0	1	2	3	4
0		31300528	541732	1528	1
1	31300528	5427200	31232	4	
2	541732	31232	6		
3	1528	4			
4	1				

Table: Number of NCRSFs with k cycles of slope (i, j) on the torus $T_{4,4}$

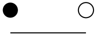
Computation for $W, H \leq 9$

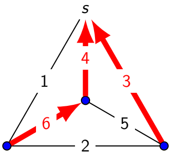
Inverse function

Placing the grains on the edges. 



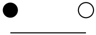
Inverse function

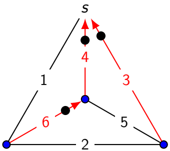
Placing the grains on the edges. 



- Orientation towards the sink

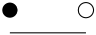
Inverse function

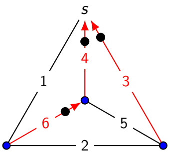
Placing the grains on the edges. 



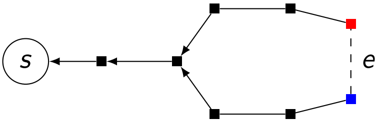
- Orientation towards the sink
- Internal: 1 grain • to the father

Inverse function

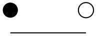
Placing the grains on the edges. 

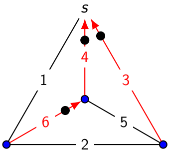


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

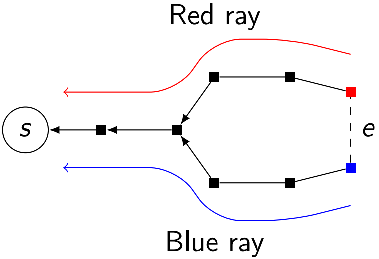


Inverse function

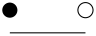
Placing the grains on the edges. 

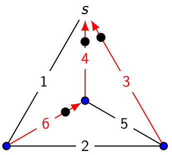


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

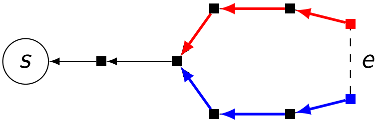


Inverse function

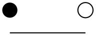
Placing the grains on the edges. 

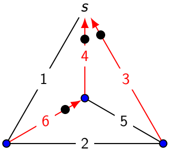


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

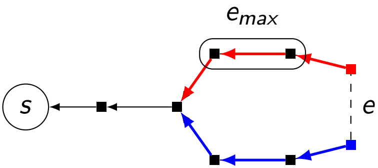


Inverse function

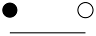
Placing the grains on the edges. 

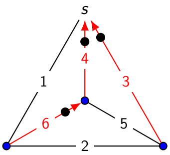


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

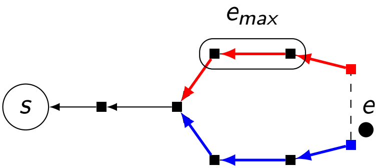


Inverse function

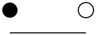
Placing the grains on the edges. 

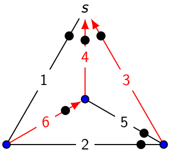


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

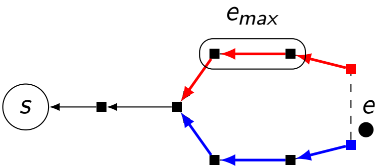


Inverse function

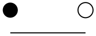
Placing the grains on the edges. 

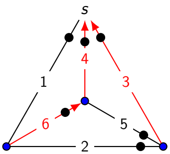


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle

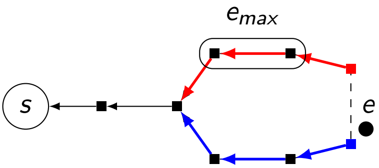


Inverse function

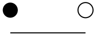
Placing the grains on the edges. 

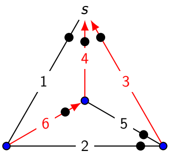


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active

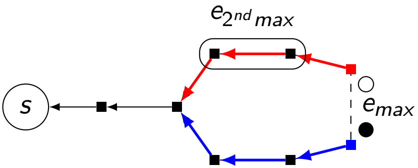


Inverse function

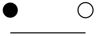
Placing the grains on the edges. 

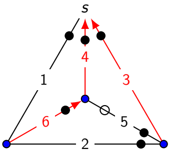


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active

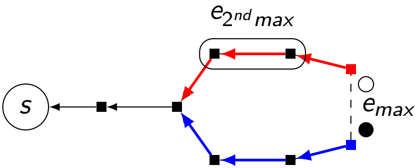


Inverse function

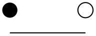
Placing the grains on the edges. 

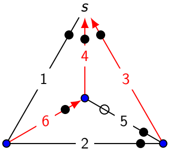


- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ○ on the other endpoint if active

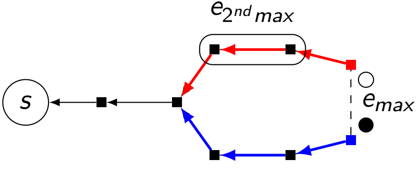


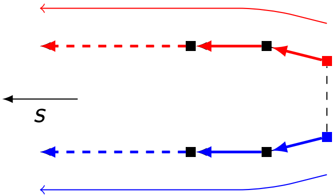
Inverse function

Placing the grains on the edges. 



- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ◦ on the other endpoint if active

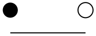


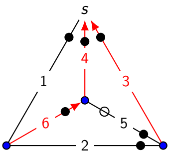


Infinite decreasing path

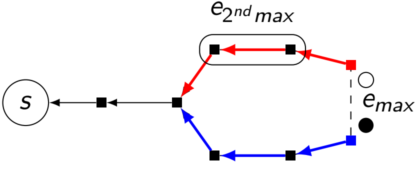
Cycles are directed such that they are globally decreasing.
Periodicity ⇒ Maximal edge at finite distance

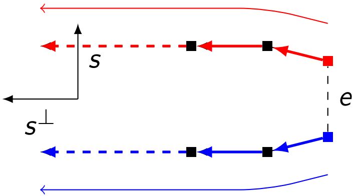
Inverse function

Placing the grains on the edges. 



- Orientation towards the sink
- Internal: 1 grain • to the father
- External: • depends on the position of the maximal edge on the fundamental cycle
- External: ∘ on the other endpoint if active





Infinite decreasing path

Cycles are directed such that they are globally decreasing.
Periodicity ⇒ Maximal edge at finite distance

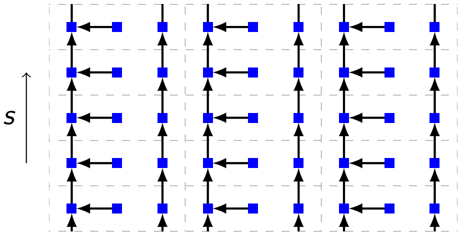
Restricted Tutte Polynomial

$$\mathcal{T}_{W\times H,\mathbf{s}}(x,y) = \sum_{T\in\mathcal{F}_{W\times H}} x^{\mathrm{int}_{\mathrm{W}\times\mathrm{H}}(T)} y^{\mathrm{ext}_{\mathrm{W}\times\mathrm{H}}(T)}$$

$e <_{\mathbf{s}} f$ if e is closer to the sink than f.

Restrictions

- On NCRSF: $\mathcal{F}_{W\times H}$.
- On the activity: on the rectangular fundamental domain $W \times H$ consider exactly $2WH$ edges.



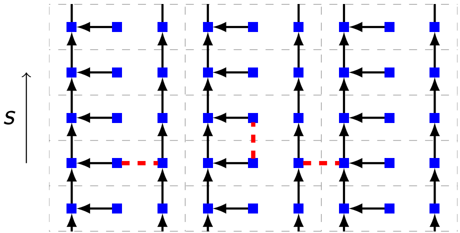
Restricted Tutte Polynomial

$$\mathcal{T}_{W\times H,\mathbf{s}}(x,y) = \sum_{T\in\mathcal{F}_{W\times H}} x^{\mathrm{int}_{\mathrm{W\times H}}(T)} y^{\mathrm{ext}_{\mathrm{W\times H}}(T)}$$

$e <_{\mathbf{s}} f$ if e is closer to the sink than f.

Restrictions

- On NCRSF: $\mathcal{F}_{W\times H}$.
- On the activity: on the rectangular fundamental domain $W \times H$ consider exactly $2WH$ edges.



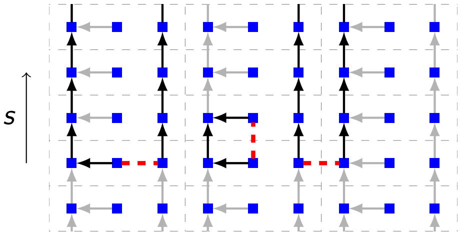
Restricted Tutte Polynomial

$$\mathcal{T}_{W\times H,\mathbf{s}}(x,y) = \sum_{T\in\mathcal{F}_{W\times H}} x^{\mathrm{int}_{\mathrm{W\times H}}(T)} y^{\mathrm{ext}_{\mathrm{W\times H}}(T)}$$

$e <_{\mathbf{s}} f$ if e is closer to the sink than f.

Restrictions

- On NCRSF: $\mathcal{F}_{W\times H}$.
- On the activity: on the rectangular fundamental domain $W \times H$ consider exactly $2WH$ edges.



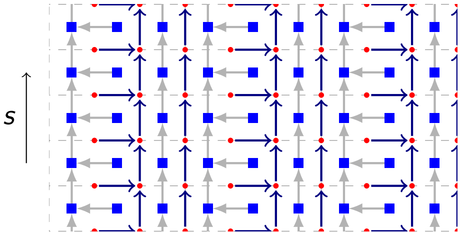
Restricted Tutte Polynomial

$$\mathcal{T}_{W\times H,\mathbf{s}}(x,y) = \sum_{T\in\mathcal{F}_{W\times H}} x^{\mathrm{int}_{W\times H}(T)} y^{\mathrm{ext}_{W\times H}(T)}$$

$e <_{\mathbf{s}} f$ if e is closer to the sink than f.

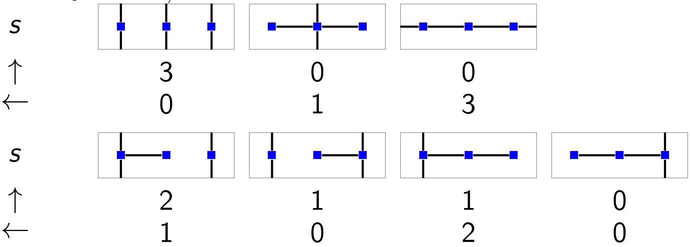
Restrictions

- On NCRSF: $\mathcal{F}_{W\times H}$.
- On the activity: on the rectangular fundamental domain $W \times H$ consider exactly $2WH$ edges.



Restricted Tutte Polynomial

External activity on $\mathcal{F}_{3,1}$:

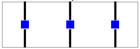
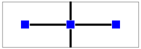
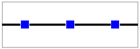
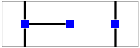
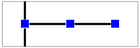
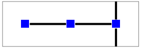


$$\mathcal{T}_{3\times 1,(0,1)}(1,y) = y^3 + 3y^2 + 6y + 7$$

$$\mathcal{T}_{3\times 1,(-1,0)}(1,y) = y^3 + 3y^2 + 6y + 7$$

Restricted Tutte Polynomial

External activity on $\mathcal{F}_{3,1}$:

s				
$\uparrow$	3	0	0	
$\leftarrow$	0	1	3	
s				
$\uparrow$	2	1	1	0
$\leftarrow$	1	0	2	0

$$\mathcal{T}_{3\times 1,(0,1)}(1,y) = y^3 + 3y^2 + 6y + 7$$

$$\mathcal{T}_{3\times 1,(-1,0)}(1,y) = y^3 + 3y^2 + 6y + 7$$

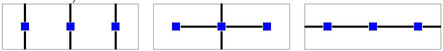
Theorem (D., Le Borgne 2018)

For any directions $\mathbf{s}, \mathbf{s}'$, $\mathcal{T}_{W\times H,\mathbf{s}}(1,y) = \mathcal{T}_{W\times H,\mathbf{s}'}(1,y)$.

Restricted Tutte Polynomial

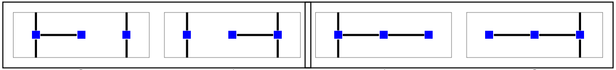
External activity on $\mathcal{F}_{3,1}$:

s



$\uparrow$	3	0	0
$\leftarrow$	0	1	3

s



$\uparrow$	2	1	1	0
$\leftarrow$	1	0	2	0

$$\mathcal{T}_{3\times 1,(0,1)}(1,y) = y^3 + 3y^2 + 6y + 7$$

$$\mathcal{T}_{3\times 1,(-1,0)}(1,y) = y^3 + 3y^2 + 6y + 7$$

Theorem (D., Le Borgne 2018)

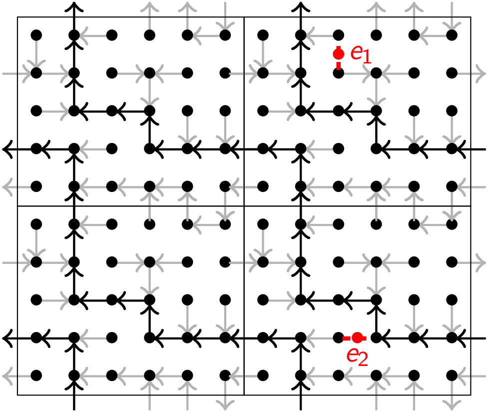
For any directions $\mathbf{s}, \mathbf{s}'$, $\mathcal{T}_{W\times H,\mathbf{s}}(1,y) = \mathcal{T}_{W\times H,\mathbf{s}'}(1,y)$.

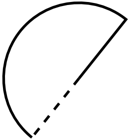
Since $\mathbb{Z}^2$ is self-dual, we have:

$$\mathcal{T}_{3\times 1,(0,1)}(x,y) = x^3y^3 + 3xy^2 + 3x^2y + 3x + 3y + 4$$

$$\mathcal{T}_{3\times 1,(-1,0)}(x,y) = x^3y^3 + 3x^2 + 3y^2 + 3xy + 3x + 3y + 1$$

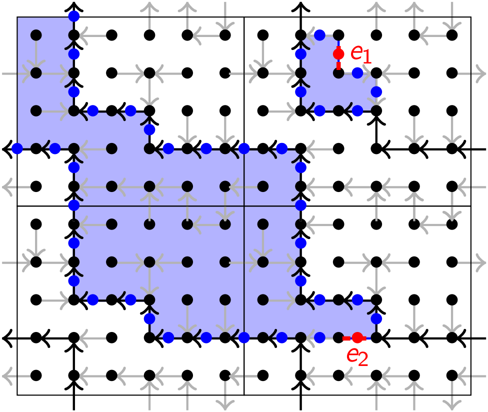
External activity

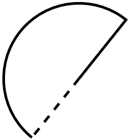




Direction of the sink

External activity

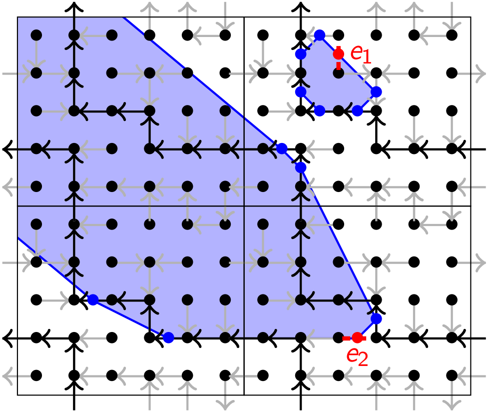


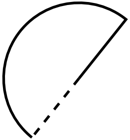


Direction of the sink

- Convex hulls of fundamental cycles.

External activity

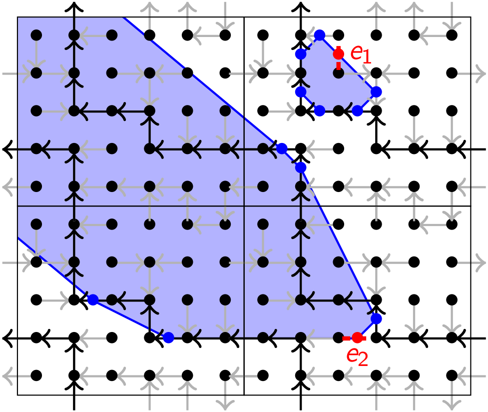


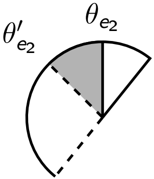


Direction of the sink

- Convex hulls of fundamental cycles.
- Active ⇒ Convex hull corner

External activity



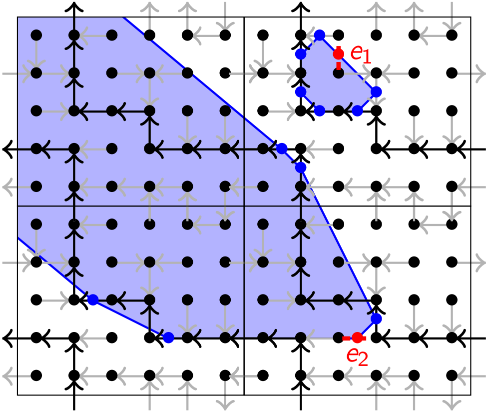


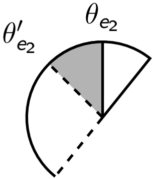
Direction of the sink

- Convex hulls of fundamental cycles.
- Active $\Rightarrow$ Convex hull corner

For each external edge e, there is an activity sector $[\theta_e, \theta'_e)$.

External activity



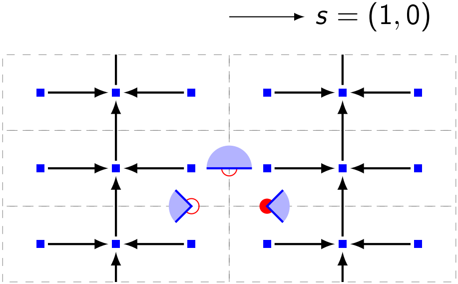


Direction of the sink

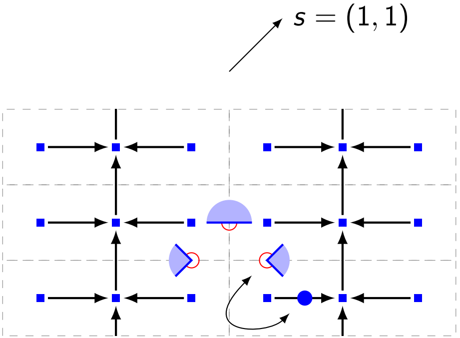
- Convex hulls of fundamental cycles.
- Active $\Rightarrow$ Convex hull corner

For each external edge e, there is an activity sector $[\theta_e, \theta'_e)$.
For any sector excluding all $(\theta_e)_e$ and $(\theta'_e)_e$, the external activity is invariant.

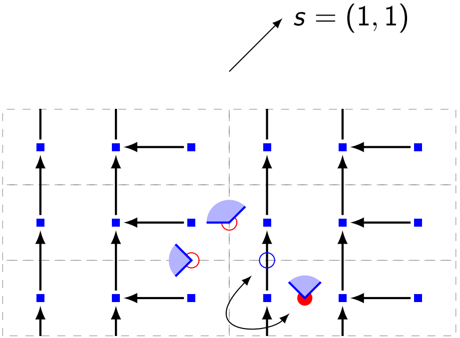
Critical pair exchange : Rotation step



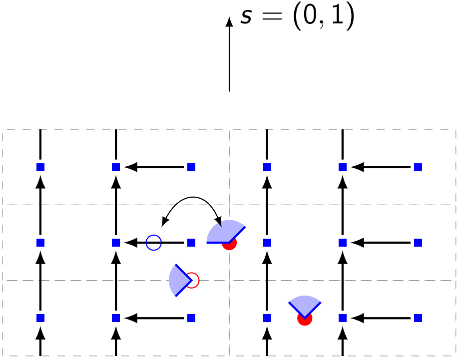
Critical pair exchange : Rotation step



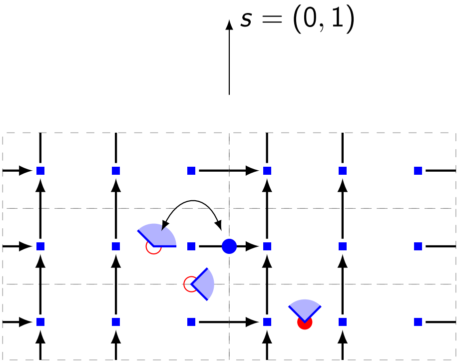
Critical pair exchange : Rotation step



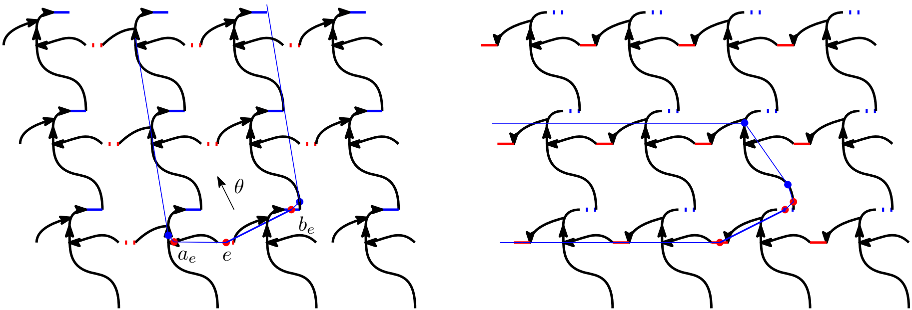
Critical pair exchange : Rotation step



Critical pair exchange : Rotation step



Critical pair exchange: changing forest slope



Checkpoint

Finite graphs	Square lattice (biperiodicity)
▷ Stable configurations	▷ Biperiodic stable configurations
▷ Dhar Criterion	▷ Weak Dhar Criterion (projective sink)
▷ Bijection between recurrent and spanning trees	▷ Bijection *recurrent* and some spanning forests of the torus
▷ Tutte polynomial	▷ Restriction of Tutte polynomial
▷ Invariant by edge exchange	▷ Distribution of external activity invariant by rotation of projective sink
▷ Symmetric for self-dual planar graphs	▷ Symmetric joint distribution of external/internal activities *changing* by rotation

Conclusion

We have

Conclusion

We have

- Weak Dhar Criterion efficient for biperiodic configurations

Conclusion

We have

- Weak Dhar Criterion efficient for biperiodic configurations
- Bijection with NCRSFs, extending the definition of biperiodic recurrent configurations

Conclusion

We have

- Weak Dhar Criterion efficient for biperiodic configurations
- Bijection with NCRSFs, extending the definition of biperiodic recurrent configurations
- Invariant distribution of external activity on NCRSFs and order given by a direction

Conclusion

We have

- Weak Dhar Criterion efficient for biperiodic configurations
- Bijection with NCRSFs, extending the definition of biperiodic recurrent configurations
- Invariant distribution of external activity on NCRSFs and order given by a direction
- Involution on NCRSFs for atomic rotation preserving this distribution

Conclusion

Perspectives

Conclusion

Perspectives

- $\mathcal{T}_{W \times H, s}(x, y)$ depends on s

Conclusion

Perspectives

- $\mathcal{T}_{W \times H, s}(x, y)$ depends on s
- Iteration of the rotation step can take several rounds before the identity

Conclusion

Perspectives

- $\mathcal{T}_{W\times H,s}(x,y)$ depends on s
- Iteration of the rotation step can take several rounds before the identity
- What about other orders ?

Conclusion

Perspectives

- $\mathcal{T}_{W \times H,s}(x, y)$ depends on s
- Iteration of the rotation step can take several rounds before the identity
- What about other orders ?
  - Experiments: periodic decreasing orders towards a direction is enough: $e <_E f \Rightarrow e + (iW, jH) <_E f + (iW, jH)$ and $\langle s, (iW, jH) \rangle > 0 \Rightarrow e + (iW, jH) <_E e$

Conclusion

Perspectives

- $\mathcal{T}_{W\times H,s}(x,y)$ depends on s
- Iteration of the rotation step can take several rounds before the identity
- What about other orders ?
  - Experiments: periodic decreasing orders towards a direction is enough: $e <_E f \Rightarrow e + (iW, jH) <_E f + (iW, jH)$ and $\langle s, (iW, jH)\rangle > 0 \Rightarrow e + (iW, jH) <_E e$
  - Only decreasing, or only periodic

Conclusion

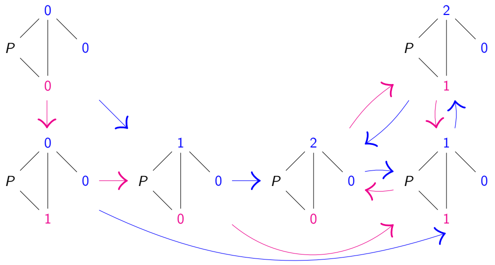
Perspectives

- $\mathcal{T}_{W\times H,s}(x,y)$ depends on s
- Iteration of the rotation step can take several rounds before the identity
- What about other orders ?
  - Experiments: periodic decreasing orders towards a direction is enough: $e <_E f \Rightarrow e + (iW, jH) <_E f + (iW, jH)$ and $\langle s, (iW, jH)\rangle > 0 \Rightarrow e + (iW, jH) <_E e$
  - Only decreasing, or only periodic
  - Anything else

THANK YOU

Markov Chain for $G = (V \cup \{S\}, E)$

- States: stable configurations on G
- Transition: Add a particle to a vertex chosen uniformly and stabilize

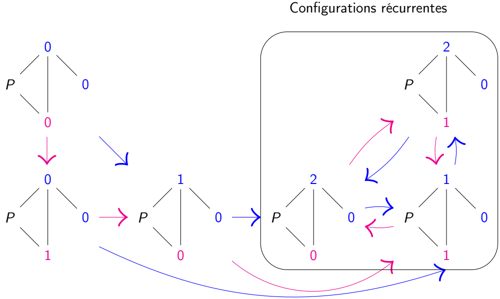


- The recurrent states are called recurrent configurations.
- The stationary distribution is uniform on the recurrent configurations.

Dhar Criterion A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)

Markov Chain for $G = (V \cup \{S\}, E)$

- States: stable configurations on G
- Transition: Add a particle to a vertex chosen uniformly and stabilize



- The recurrent states are called recurrent configurations.
- The stationary distribution is uniform on the recurrent configurations.

Dhar Criterion A stable configuration is recurrent if and only if adding a grain to each neighbor of the sink, and stabilizing result to the same configuration. (fixed point)

Close to [Pegden and Smart, 2017]

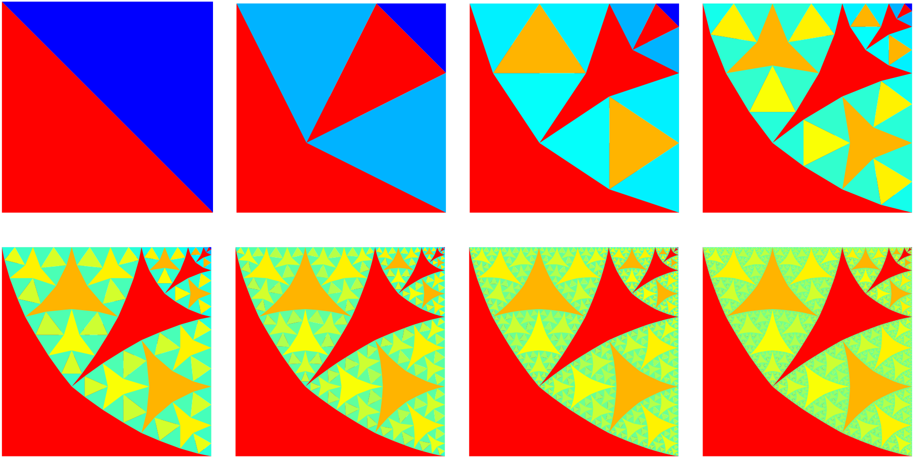


Figure: Each non blue zone is described by a quadratic form. [arxiv:1708.09432]

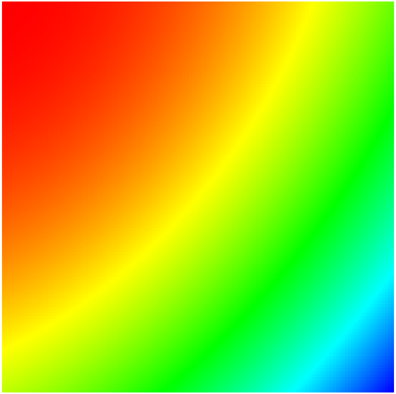
Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

The number of topples is:

$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
$$= (c+a)x^2 + 2bxy + (c-a)y^2$$

Sample for $M(0.25, 0.875, 2.125)$



Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a, b, c) = \begin{pmatrix} c + a & b \\ b & c - a \end{pmatrix}$$

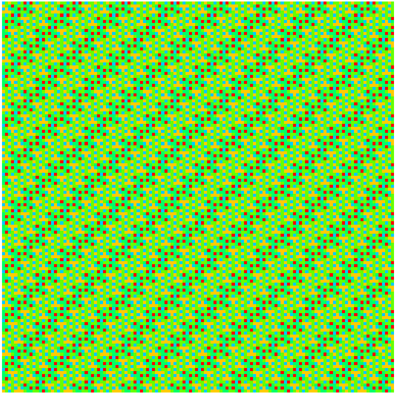
The number of topples is:

$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a, b, c)\mathbf{x} \right\rceil$$
$$= (c + a)x^2 + 2bxy + (c - a)y^2$$

Then number of grains is

$$\Delta h(\mathbf{u}) = \sum_{\mathbf{v} \sim \mathbf{u}} h(\mathbf{v}) - h(\mathbf{u}).$$

Sample for $M(0.25, 0.875, 2.125)$



Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

The number of topples is:

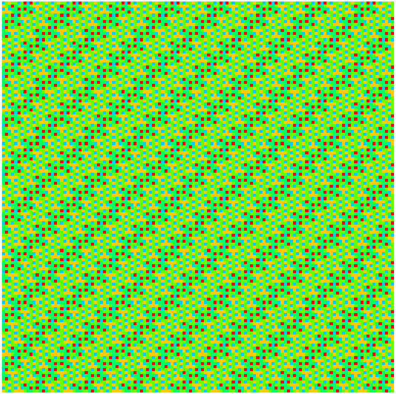
$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
$$= (c+a)x^2 + 2bxy + (c-a)y^2$$

Then number of grains is

$$\Delta h(\mathbf{u}) = \sum_{\mathbf{v}\sim\mathbf{u}} h(\mathbf{v}) - h(\mathbf{u}).$$

- It's *periodic* for $a, b, c \in \mathbb{Q}$

Sample for $M(0.25, 0.875, 2.125)$



Quadratic forms for periodic zones [Levine, Pegden, Smart 2012]

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

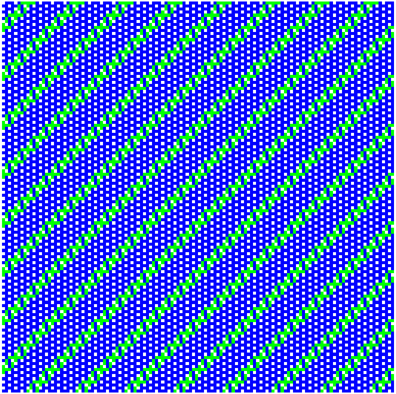
The number of topples is:

$$h(\mathbf{x}) = \left\lceil \frac{1}{2}\mathbf{x}^t M(a,b,c)\mathbf{x} \right\rceil$$
$$= (c+a)x^2 + 2bxy + (c-a)y^2$$

Then number of grains is

$$\Delta h(\mathbf{u}) = \sum_{\mathbf{v}\sim\mathbf{u}} h(\mathbf{v}) - h(\mathbf{u}).$$

Sample for $M(0.25, 0.875, 2.125)$



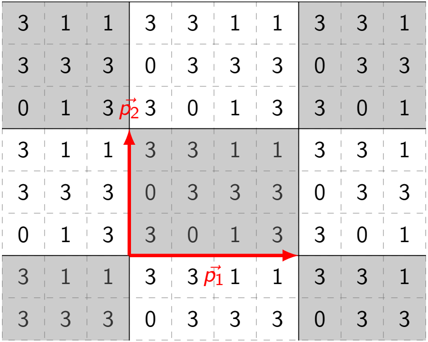
- It's *periodic* for $a, b, c \in \mathbb{Q}$
- But it may be negative and/or unstable!

It may be stabilized without changing density of grains.

A definition of recurrence for periodic stable configurations

Pattern + two dimensional period $(\vec{p_1}, \vec{p_2})$.

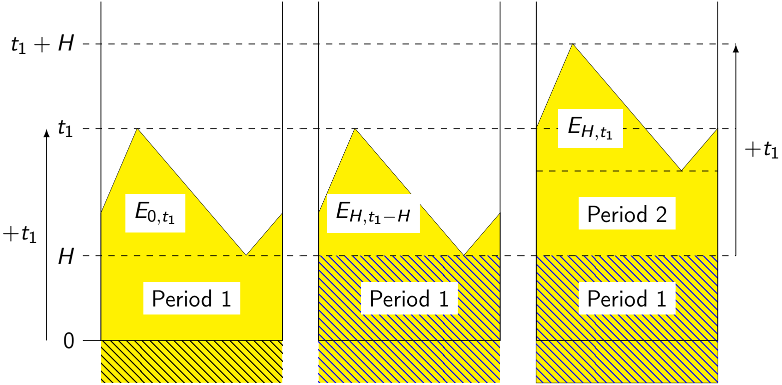
$$\forall \mathbf{x} \in \mathbb{Z}^2 u(\mathbf{x}) = u(\mathbf{x} + \vec{p_1}) = u(\mathbf{x} + \vec{p_2})$$



Periodicity on $-\mathbf{s}$

Lemme

If a periodic configuration is recurrent, then there exists a position $y = t_1$ for which all vertices of the first period are toppled.



We have $Period1 \subset E_{0,t_1} \Rightarrow E_{0,t_1} = E_{H,t_1-H}$ and $v \in E_{0,t_1} \Rightarrow v + H\vec{y} \in E_{H,t_1}$. Then $E_{H,t_1} \supset Period2$.